

Space-time crystal and space-time group symmetry

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Crystal structures and the Bloch theorem play a fundamental role in condensed matter physics. We propose “space-time” crystals exhibiting the general intertwined space-time periodicities in $D+1$ dimensions, which include the Floquet lattice systems as a special case. Their crystal symmetry structures are described by “space-time” groups. Compared to space and magnetic groups, they are augmented by “time-screw” rotations and “time-glide” reflections involving fractional time translations. A complete classification of the 13 space-time groups in 1+1D is performed. Kramers-type degeneracy can arise from space-time symmetries without the half-integer spinor structure, which constrains the winding number patterns of spectral dispersions. In 2+1D, non-symmorphic space-time symmetries enforce spectral degeneracies, leading to protected Floquet semi-metal states. Our work provides a general framework for further studying topological properties of the $D+1$ dimensional space-time crystals.

The fundamental concept of energy bands in crystals based on the Bloch theorem lays the foundation of modern condensed matter physics. In the past decade, studies on band structure symmetry and topology lead to important discoveries of the topological insulating state, topological superconductivity, and the Weyl semi-metal state^{1–3}. More recently, periodically driven systems have also attracted increasing interests. Periodic driving provides a new route to engineer topological states in systems originally topologically trivial in the absence of driving. Such possibilities have been explored in various systems, including the irradiated graphene^{4,5}, semiconducting quantum wells⁶, dynamically modulated cold atom optical lattices⁷, and photonic systems^{8,9}. For driven systems with a temporal period T , Bloch bands are replaced by Bloch-Floquet bands, which are periodic in energy space since the quasi-energy is only conserved module $2\pi/T$. This feature further enriches topological band structures^{10–12}. For non-interacting systems, this extra periodicity leads to the dynamically generated Majorana modes in 1D¹³, and anomalous edge states with zero Chern number in 2D¹⁴. Topological classifications for interacting Floquet systems have also been investigated^{15–19}.

Symmetry plays a fundamental role in analyzing topological properties of periodically driven systems. However, previous studies mostly treat the temporal periodicity separated from the spatial one. In fact, the driven system can exhibit much richer symmetry structures than a simple direct product of symmetries in space and time domains. In particular, a temporal translation at a *fractional* period can be combined with space group symmetries to form novel space-time intertwined symmetries, which, to the best of our knowledge, have not yet been fully explored. For static lattices, the intrinsic connections between the space-group symmetries and physical properties, especially the topological phases, have been extensively studied^{20–24}. Therefore, it is expected that the intertwined space-time symmetries could also protect novel trivial properties of the driven system, regardless of microscopic details.

In this article, we generalize the concept of Floquet-

Bloch lattices to space-time crystals, which exhibit intertwined space-time symmetries. Space-time crystals exhibit the periodicities characterized by $D+1$ linearly independent basis vectors, which are space-time mixed in general. The usual Floquet-Bloch systems are a special case exhibiting separate spatial and temporal periodicities. The full discrete space-time symmetries of space-time crystals form groups – dubbed “space-time” groups, which are generalizations of space groups for static crystals by including “time-screw” and “time-glide” operations. A complete classification of the 13 space-time groups in 1+1 D is performed, and their constraints on band structure winding numbers are studied. In 2+1 D, the non-symmorphic space-time symmetry operations, similar to their static space-group counterparts, lead to spectral degeneracies for periodically driven systems, even when the instantaneous spectra are gapped at any given time t .

Space-time lattice – We consider the time-dependent Hamiltonian $H = P^2/(2m) + V(\mathbf{r}, t)$ in the $D+1$ dimensional space-time. $V(\mathbf{r}, t)$ exhibits the intertwined discrete space-time translational symmetry as

$$V(\mathbf{r}, t) = V(\mathbf{r} + \mathbf{u}^i, t + \tau^i), \quad i = 1, 2, \dots, D+1, \quad (1)$$

where $(\mathbf{u}^i, \tau^i) = a^i$ is the primitive basis vector of the space-time lattice. Eq. 1 extends the usual Floquet-Bloch lattice with the separated spatial and temporal periodicities to the more general case with space-time mixed primitive vectors. In general, the space-time primitive unit cell is not a direct product between spatial and temporal domains. There may not even exist spatial translational symmetry at any given time t , nor temporal translational symmetry at any spatial location \mathbf{r} . Consequently, the frequently used time-evolution operator $U(T)$ of one period for the Floquet problem generally does not apply. The reciprocal lattice is spanned by the momentum-energy basis vectors $b^i = (\mathbf{G}^i, \Omega^i)$ defined through $b^i \cdot a^j = \sum_{m=1}^D G_m^i u_m^j - \Omega^i \tau^j = 2\pi \delta^{ij}$. The $D+1$ dimensional momentum-energy Brillouin zone (MEBZ) may also be momentum-energy mixed.

We generalize the Floquet-Bloch theorem for the time-dependent Schrödinger equation $i\hbar \partial_t \psi(\mathbf{r}, t) =$

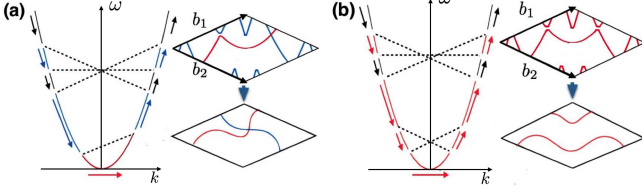


FIG. 1. Folding the band dispersions of the 1+1 D space-time crystal into the 1st rhombic MEBZ in the weak lattice limit. The momentum-energy reciprocal lattice vectors of nonzero V_B 's are represented by dashed lines. The low-energy part of the free dispersion curve evolves to closed loops. (a) Two loops with the winding numbers $\mathbf{w}_r = (1, 0)$ (red) and $\mathbf{w}_b = (0, 1)$ (blue). (b) An extra nonzero V_G connects two loops in (a) forming a new one with $\mathbf{w} = \mathbf{w}_r + \mathbf{w}_b$.

$H(\mathbf{r}, t)\psi(\mathbf{r}, t)$. Due to the space-time translation symmetry, the lattice momentum-energy vector $\kappa = (\mathbf{k}, \omega)$ is conserved. Only the κ vectors inside the first MEBZ are non-equivalent, and the κ vectors outside are equivalent to those inside up to integer reciprocal lattice vectors. The states characterized by κ take the form of

$$\psi_{\kappa, m}(\mathbf{r}, t) = e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_m t)} u_m(\mathbf{r}, t), \quad (2)$$

where m marks different states sharing the common κ . $u_m(\mathbf{r}, t)$ processes the same space-time periodicity as $H(\mathbf{r}, t)$, and is expanded as $u_m = \sum_B c_{m, B} e^{i(\mathbf{G} \cdot \mathbf{r} - \Omega t)}$ with $B = (\mathbf{G}, \Omega)$ taking all the momentum-energy reciprocal lattice vectors. The eigen-frequency ω_m is determined through the eigenvalue problem defined as

$$\sum_{B'} \{[-\Omega + \varepsilon_0(\mathbf{k} + \mathbf{G})]\delta_{B, B'} + V_{B-B'}\} c_{m, B'} = \omega_m c_{m, B}, \quad (3)$$

where $\varepsilon_0(\mathbf{k})$ is the free dispersion, and V_B is the momentum-energy Fourier component of the space-time lattice potential $V(\mathbf{r}, t)$. The dispersion based on Eq. 3 is represented by a D -dimensional surface in the MEBZ which is a $D+1$ dimensional torus.

The energy band structures of the space-time lattice exhibit novel features different from those with separated spatial and temporal periodicities. For simplicity, below we use the 1+1 D case to investigate the symmetry and topological properties of the space-time crystal.

Dispersion winding numbers – The dispersion relation $\omega(k)$ forms closed loops in the 2D toroidal MEBZ, each of which is characterized by a pair of winding numbers $\mathbf{w} = (w_1, w_2)$. Compared to the static lattice case in which the band dispersion only winds around the momentum direction, here $\omega(k)$ is typically not single-valued and its winding patterns are much richer. The dispersions in the limit of a weak space-time potential $V(x, t)$ with a rhombic MEBZ are illustrated in Fig. 1 (a) and (b). When folded into the MEBZ, the free dispersion curve $\varepsilon(k)$ can cross at general points not just on high symmetry ones. A crossing point corresponds to

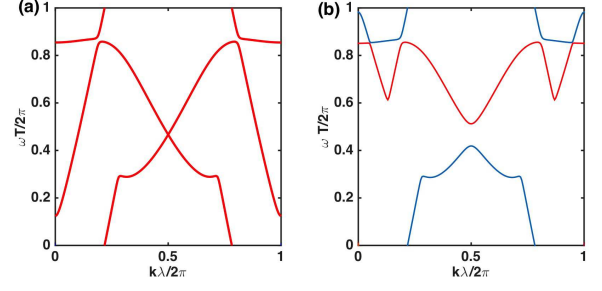


FIG. 2. (a) The Floquet-Bloch band spectrum with the space-time lattice potential possessing the glide time-reflection symmetry g_t . When applied to the states with $\kappa_x = \pi/\lambda$, g_t becomes a Kramers symmetry protecting the double-degeneracy. (b) Lifting the Kramers degeneracy by adding a glide time-reflection symmetry breaking term.

two equivalent momentum-energy points related by a reciprocal vector G before folding. When $V_G \neq 0$, the crossing is avoided by forming a gap at the magnitude of $2|V_G|$. The winding directions of the dispersion loops are generally momentum-energy mixed. Furthermore, different momentum-energy reciprocal lattice vectors can cross each other, leading to composite loops winding around the MEBZ along both directions as shown in Fig. 1 (b). The total number of states at each \mathbf{k} is invariant with lattice potential, then crossing can only split along the ω -direction and $d\omega/dk$ is always finite. Consequently, trivial loops with the winding numbers $(0, 0)$ are forbidden, while all other patterns (w_1, w_2) are possible in general.

The intertwined space-time lattice symmetries besides translations can protect crossings and impose further constraints on band structures. Consider a 1+1 D crystal structure with the space-time unit cell as the direct product of spatial and temporal periods λ and T , respectively. Its first MEBZ is also a direct-product as $[-\pi/\lambda, \pi/\lambda] \otimes [-\pi/T, \pi/T]$. We further assume the system is invariant under a combined time-reversal transformation followed by the translation of a half spatial period. This operation denoted as g_t can be viewed as a “glide time-reflection” defined as $g_t(x, t) = (x + \frac{1}{2}\lambda, -t)$. Its operation on the Hamiltonian is defined as $g_t^{-1} H g_t = H^*(g_t(x, t))$. The corresponding transformation M_{g_t} on the Bloch-Floquet wavefunction $\psi_\kappa(x, t)$ of Eq. 2 is anti-unitary defined as $M_{g_t} \psi_\kappa = \psi_\kappa^*(g_t^{-1}(x, t))$. The glide time-reflection leaves the line of $\kappa_x = \pi/\lambda$ in the MEBZ invariant. M_{g_t} becomes a Kramers symmetry for states along this line,

$$M_{g_t}^2 \psi_\kappa = \psi_\kappa(x - \lambda, t) = e^{-i\kappa_x \lambda} \psi_\kappa = -\psi_\kappa, \quad (4)$$

which arises purely from the space-time crystal symmetry without involving the half-integer spinor structure. It protects the double degeneracy of the momentum-energy quantum numbers of ψ_κ and $M_{g_t} \psi_\kappa$. Hence the crossing at $\kappa_x = \pi/\lambda$ cannot be avoided and the dispersion winding numbers along the momentum direction must

be even.

As a concrete example, we study a crystal potential with the above spatial and temporal periodicities, $V(x, t) = V_0 \left(\sin \frac{2\pi}{T} t \cos \frac{2\pi}{\lambda} x + \cos \frac{2\pi}{T} t \right)$. Except the glide time-reflection symmetry, it does not possess other space-time symmetries. Its Bloch-Floquet spectrum is calculated based on Eq. 3, and a representative dispersion loop is plotted in the MEBZ shown in Fig. 2 (a). The crossing at $\kappa_x = \pi/\lambda$ is protected by the glide time-reflection symmetry giving rise to a pair of Kramers doublet. As a result, the winding number of this loop is $\mathbf{w} = (w_x, w_t) = (2, 0)$. If a glide time-reflection symmetry breaking term $\delta V = V'_0 \cos(\frac{2\pi}{\lambda} x)$ is added into the crystal potential, the crossing is avoided due to the Kramers symmetry breaking as shown in Fig. 2 (b). Consequently, the dispersion splits into two loops, both of which exhibit the winding number $(1, 0)$.

Space-time group – We propose the concept of “space-time” group for a full description of the symmetry properties of the $D + 1$ dimensional space-time crystal structures. It not only includes space group and magnetic group transformations in the D -spatial dimensions, but also is extended to include operations involving fractional translations along the time-direction. Since space and time are non-equivalent in the Schrödinger equation, space-time rotations are not allowed except the 2-fold case. For a symmetry operation Γ of the space-time crystal, its operation on (\mathbf{r}, t) is defined as,

$$\Gamma(\mathbf{r}, t) = (R\mathbf{r} + \mathbf{u}, st + \tau), \quad (5)$$

where R is a D -dimensional point group operation, $s = \pm 1$ and $s = -1$ indicates time-reflection, *i.e.*, time-reversal, and $(\mathbf{u}, \tau) = \sum_i m_i a^i$ represents a space-time translation with m_i either integers or fractions. At $\tau = 0$, Γ is reduced to a space group or magnetic group operation according to $s = \pm 1$, respectively. At $\tau \neq 0$, when (\mathbf{u}, τ) contains fractions of a^i , new symmetry operations arise due to the dynamic nature of the crystal potential, including the “time-screw” rotation and “time-glide” reflection, which are a spatial rotation or a reflection followed by a fractional time translation, respectively. The operation of Γ on the Hamiltonian is defined as $\Gamma^{-1}H(\mathbf{r}, t)\Gamma = H(\Gamma(\mathbf{r}, t))$, or, $\Gamma^{-1}H(\mathbf{r}, t)\Gamma = H^*(\Gamma(\mathbf{r}, t))$ for $s = \pm 1$, respectively. Correspondingly, the transformation M_Γ on the Bloch-Floquet wavefunctions $\psi_\kappa(\mathbf{r}, t)$ is $M_\Gamma \psi_\kappa = \psi_\kappa(\Gamma^{-1}(\mathbf{r}, t))$, or, $\psi_\kappa^*(\Gamma^{-1}(\mathbf{r}, t))$ for $s = \pm 1$, respectively.

As a concrete example, we present a complete classification of the space-time groups for 1+1 D space-time crystal structures. Due to the non-equivalence between spatial and temporal directions, there are no square and hexagonal space-time crystal systems. The point-group like operations are isomorphic to D_2 , including reflection m_x , time reversal m_t , and their combination $m_x m_t$, *i.e.*, the 2-fold space-time rotation. Consequently, only two space-time crystal systems are allowed – oblique and orthorhombic. In addition to g_t , symmetries involving fractional translations also include the “time glide reflection”

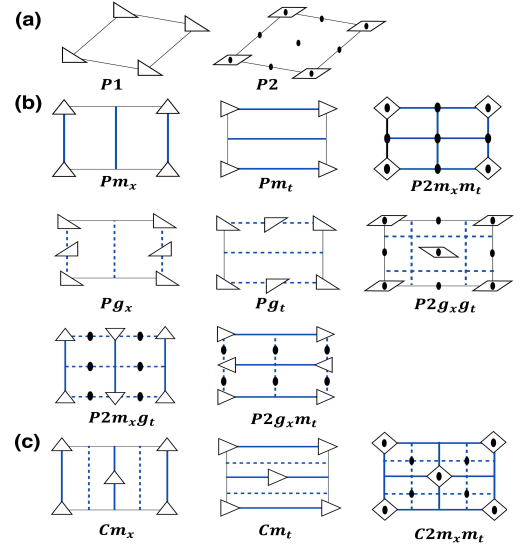


FIG. 3. The classification of 13 space-time groups in 1+1D and the associated lattice configurations. The solid oval marks the 2-fold space-time axis, and the parallelogram mean the 2-fold axis without reflection symmetries. The thick solid and dashed lines represent reflection and glide-reflection axes, respectively. Configurations of triangles and the diamond denote the local symmetries under reflections. (a) The oblique lattices with and without 2-fold axes. Their basis vectors are generally space-time mixed. The primitive (b) and centered (c) orthorhombic lattices: According to their reflection and glide reflection symmetries, they are classified to 8 groups in (b), and 3 groups in (c).

tion” g_x - spatial reflection followed by a fractional time-translation. The space-time crystal structures marked with m_x and m_t , or those with g_x and g_t , should be different, respectively.

The above 1+1 D space-time symmetries give rise to 13 space-time groups in contrast to the 17 wallpaper space groups characterizing the 2D static lattices. The oblique Bravais lattice is simply monoclinic, while the orthorhombic ones include both the primitive and centered Bravais lattices. The monoclinic lattice gives rise to two different crystal structures with and without the 2-fold space-time axes, whose space-time groups are denoted by $P_{1,2}$, respectively, as shown in Fig. 3 (a). For the primitive orthorhombic lattices, the associated crystal structures can exhibit the point-group symmetries m_x and m_t , and the space-time symmetries g_t and g_x . Their combinations give rise to crystal structures with 8 space-time group symmetries denoted as Pm_x , Pm_t , $P2m_x m_t$, Pg_x , Pg_t , $P2g_x g_t$, $P2m_x g_t$, $P2g_x m_t$, respectively, as shown in Fig. 3 (b). Four of them possess the 2-fold space-time axes as indicated by “2” in their symbols. For the centered orthorhombic Bravais lattices, 3 crystal structures exist with space-time groups denoted as Cm_x , Cm_y , and $C2m_x m_t$, respectively, as shown in Fig. 3 (c). They all exhibit glide-reflection symmetries, and the last one possesses the 2-fold space-time axes as well. Two unit cells

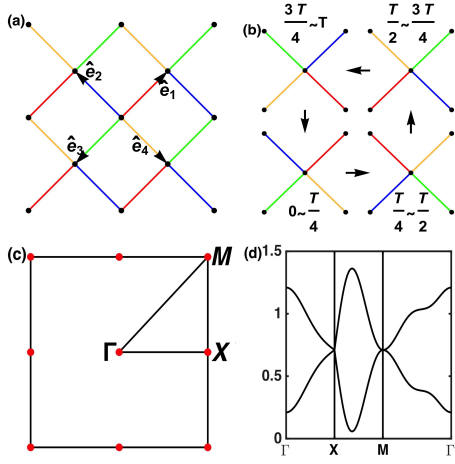


FIG. 4. (a) The 2+1 D space-time lattice structure of the Hamiltonian Eq. 7. The bond directions are marked as $\vec{e}_{1,3} = \pm\frac{1}{2}(\hat{x} + \hat{y})$, $\vec{e}_{2,4} = \mp\frac{1}{2}(\hat{x} - \hat{y})$. (b) The time-dependent hopping pattern rotates 90° every one quarter period. The bonding strengths $w_{\vec{e}_i}(t)$ of the R , B , G and Y bonds equal 0.2, 3, -3.2 , and 0.5, respectively. (c) The momentum Brillouin zone with high symmetry points $\Gamma = (0,0)$, $M = (\pm\pi, \pm\pi)$, and $X = (0, \pm\pi)$ and $(\pm\pi, 0)$. (d) The dispersions along the cuts from Γ to X to M to Γ . Two-fold degeneracies appear at X and M .

are plotted for the centered lattices to show the full symmetries explicitly, and their primitive basis vectors are actually space-time mixed.

The 2+1 D space-time crystals. - The classifications of the space-time groups in more than 1+1 D are generally complicated. The 2+1D case is analyzed in Supplementary Material, whose structures are further enriched by spatial rotations and time-screw rotations.

Below we show that the space-time group operations protect robust spectrum degeneracies and lead to the Floquet semi-metal state by using a 2+1D example. Consider the case that the space-time little group of the momentum \mathbf{k} contains two non-symmorphic space-time group operations $g_{1,2}$ satisfying

$$g_1 g_2 = T(\mathbf{u}) g_2 g_1, \quad \text{and}, \quad \mathbf{k} \cdot \mathbf{u} = 2\pi p/q, \quad (6)$$

where $T(\mathbf{u})$ is a space translation of integer lattice vectors, and p and q coprime. We find that the Bloch-Floquet wavefunctions exhibit a q -fold degeneracy at the momentum-energy vector $\kappa = (\mathbf{k}, \omega)$ proved as follows. Since g_1 belongs to the little group, $\psi_\kappa(\mathbf{r}, t)$ can be chosen to satisfy $M_{g_1} \psi_{\kappa,1} = \mu \psi_{\kappa,1}$, then $\psi_\kappa, M_{g_2} \psi_\kappa, M_{g_2}^2 \psi_\kappa, \dots, M_{g_2}^{q-1} \psi_\kappa$ are the common Bloch-Floquet eigenstates sharing the same κ but exhibiting a set of different eigenvalues of g_1 as $\eta, \mu\eta, \mu\eta^2, \dots, \mu\eta^{q-1}$ with $\eta = e^{i\pi p/q}$. Then they are orthogonal to each other forming a q -fold degeneracy. Compared to the case of non-symmorphic space group protected degeneracy^{21,22,24}, here $g_{1,2}$ are space-time operations for a dynamic space-time crystal.

We employ a 2+1 D tight-binding space-time model as

an example to illustrate the above protected degeneracy. A snapshot of the lattice is depicted in Fig. 4 (a), which consists of two sublattices: The A -type sites are with integer coordinates (i, j) , and each A -site emits four bonds along \vec{e}_i to its four neighboring B sites at $(i \pm \frac{1}{2}, j \pm \frac{1}{2})$. The space-time Hamiltonian within the period T is

$$H(t) = - \sum_{\vec{r} \in A, \vec{r} + \frac{a}{2} \vec{e}_i \in B} \{w_{\vec{e}_i}(t) c_{\vec{r}}^\dagger d_{\vec{r} + \frac{a}{2} \vec{e}_i} + h.c.\}, \quad (7)$$

where a is the distance between two nearest A sites, and $w_{\vec{e}_i}(t)$'s are hopping amplitudes with different strengths. Their time-dependence is illustrated in Fig. 4 (b): Within each quarter period, $w_{\vec{e}_i}$ does not vary, and their pattern rotates 90° after every $T/4$. At each given time, the lattice possesses a simple 2D space group symmetry $p2111$, which only includes two-fold rotations around the AB -bond centers without reflection and glide-plane symmetries. For example, the rotation R_π around $(\frac{a}{4}, \frac{a}{4})$ transforms the coordinate $(x, y, t) \rightarrow (\frac{a}{2} + x, \frac{a}{2} + y, t)$. In addition, there exist “time-screw” operations, say, an operation S defined as a rotation around an A -site $(0, 0)$ at 90° followed by a time-translation at $T/4$, which transforms $(x, y, t) \rightarrow (y, -x, t + \frac{T}{4})$. R_π and S are generators of the space-time group for Eq. 7. Since S is a time-screw rotation, this space-time group is non-symmorphic. It is isomorphic to the 3D space-group $I4_1$, while its 2D space subgroup $p2111$ is symmorphic. We have checked that, for a static Hamiltonian taking any of the bond configuration in Fig. 4 (b), the energy spectra are fully gapped. However, the non-symmorphic space-time group gives rise to spectral degeneracies. Its momentum Brillouin zone is depicted in Fig. 4 (c). The space-time little group of the M -point (π, π) contains both R and S satisfying $RS = T(a\hat{y})SR = -SR$. Similarly, the X -point $(\pi, 0)$ is invariant under both R and S^2 satisfying $RS^2 = T(a\hat{x} + a\hat{y})S^2R = -S^2R$. Hence, the Floquet eigen-energies are doubly degenerate at M and X -points as shown in Fig. 4 (d), showing a semi-metal structure.

In conclusion, we have studied a novel class of $D+1$ dimensional crystal structures exhibiting the general space-time periodicities. Their momentum-energy Brillouin zones are $D+1$ dimensional torus and are typically momentum-energy entangled. The band dispersions exhibit non-trivial windings around the momentum-energy Brillouin zones. The space-time crystal structures are classified by the “space-time” symmetry groups, which extend the space group for static crystal structures by incorporating “time-screw” rotations and “time-glide” reflections. In 1+1D, a complete classification of the 13 space-time groups is performed. Space-time symmetries give rise to novel Kramers degeneracy independent of the half-integer spinor structure. The non-symmorphic space-time group operations lead to protected spectral degeneracies for the 2+1 D space-time lattices.

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Appendix A: Complete classification of 2+1D space-time groups

The space-time groups in $d + 1$ dimensions $G(d, 1)$ are discrete subgroups of the Euclidean group $E_d \otimes E_1$. Each space-time group is constructed from a Bravais lattice M constituted of $d + 1$ space-time mixing discrete translations, and a magnetic point groups (MPG) in d dimensions $G_m(d)$ that leaves M invariant. The classification scheme of space-time groups is hierarchical. All space-time groups constructed from the same M and a corresponding MPG G_m form an arithmetic crystal class (ACC). It is worth noting that, in some cases, one MPG has several ways acting on M depending on the relative orientation between the symmetry operation and the Bravais lattice, giving rise to more than one ACCs. All ACCs with the same MPG form a geometric class (GCC), labeled by a set of Bravais lattices $\{M\}$ and one MPG. All GCCs with the same $\{M\}$ form a crystal system, labeled by $\{M\}$ and a set of MPGs $\{G_m\}$ which *only* act on Bravais lattices within $\{M\}$. Such hierarchy is similar to the conventional classification scheme²⁵.

Given an ACC labeled by M and G_m , the classification of all the space-time groups can be done using group cohomology theory, similar to the classification of the space groups²⁶. A space-time group G is the group extension of an abelian discrete translation group of M by the magnetic point group G_m , described by the following exact sequence,

$$1 \rightarrow M \rightarrow G \rightarrow G_m \rightarrow 1 \quad (\text{A1})$$

Such group extension can be constructed by associating each element g in $G_m(d)$ a fractional translation $c(g) \in T(d + 1)/M$, where $T(d + 1)$ is the group of continuous translations in $d + 1$ dimensions. The map c needs to satisfy

$$c(1) = 0, \quad c(g_1 g_2) = c(g_1) + g_1 c(g_2), \quad (\text{A2})$$

so that the elements $(c(g), g)$ form a group. In order to classify all the space-time groups within the ACC, a key observation is that all maps c 's themselves form an abelian group. Given two distinct assignments c_1 and c_2 , one can check that their product $c_1 \cdot c_2$, defined as,

$$c_1 \cdot c_2(g) = c_1(g) + c_2(g) \quad (\text{A3})$$

satisfies Eq. A2 as well. This group is denoted as $Z^1(G_m, T/M)$. However, not all the elements in $Z^1(G_m, T/M)$ are corresponding to distinct types of space-time groups. Without specifying the equivalence relations, the group $Z^1(G_m, T/M)$ is not finite. An obvious condition is that space groups that related by origin shifting are of the same type. This is to say that the map with the following form,

$$c_u(g) = gu - u \quad (\text{A4})$$

for any origin shifting vector $u \in T/M$, should be identified with the trivial map $c(g) = 0$. The map

with the structure in Eq. A4 also form a group $B^1(G_m, T/M)$. We are interested in the quotient group $Z^1(G_m, T/M)/B^1(G_m, T/M)$, which is the one-dimensional cohomology group of G_m with coefficients in T/M , denoted as $H^1(G_m, T/M)$.

Each element of $H^1(G_m, T/M)$ is corresponding to a type of space-time groups. Since $H^1(G_m, T/M)$ is finite, we have a finite list of space-time groups within the ACC. The trivial element of $H^1(G_m, T/M)$ is that $c(g) = 1$ for all g in the MPG, and the corresponding space-time group is the semi-direct product of the Bravais lattice M and the MPG $G_m(d)$, called symmorphic space-time group. Each ACC only contains one symmorphic space-time group. The other elements in $H^1(G_m, T/M)$ are corresponding to the nonsymmorphic space-time groups in which there is not a single site that the whole symmetry group G_m can be realized.

However, not all the elements of the cohomology group $H^1(G_m, T/M)$ lead to distinct type of the space-time groups neither. For example, two elements in $H^1(G_m, T/M)$ related by global rotation should be identified. Here we invoke the second equivalence relation, which generalizes the one used in classifying space group. All elements in $H^1(G_m, T/M)$ related by linear transformations ρ are identified, where ρ leaves *all* the MPGs within the given crystal system unchanged. According to the definition, the second equivalent relation depending on the crystal system that contains the ACC.

In the main text, we list the classification of the space-time groups in 1+1 dimensions as a proof of concept. In this section, we focus on the case of $d = 2$. The case of $d = 3$ is left for future study. In 2 dimensions, the conventional 10 crystallographic point groups, combining with the time reversal symmetry, can be enriched to 31 MPGs listed in Table. I. By combining the MPGs with three independent discrete translations in $T(3)$, we obtain 7 crystal systems, 14 Bravais lattices and 275 space-time groups. The relation between space-time crystal systems, MPGs, Bravais lattices and space-time lattices are summarized in Table II. We adopt the terminology mostly from 3D crystallography²⁷ when it is possible.

Before going into the details of each space-time group, we emphasize that the classification of the space-time group $G(2, 1)$ is different from the usual space group $G(3)$ ²⁷. The differences already appear on the level of crystal systems. The familiar cubic crystal system from space group classification is absent in 2+1 D, because its symmetry operation necessarily involves four-fold space-time rotation, which is not an isometry in $E_2 \otimes E_1$. On the other hand, there are two crystal systems, r-monoclinic and t-monoclinic, respectively, corresponding to the 3D monoclinic crystal system. The primitive unit cell of the two lattice are plotted in Fig. 5. The maximal MPG of the two crystal systems are inequivalent. The first case is $21'$, generated by spatial rotation R_π and time reversal m_t , while the second case is $m'm_2$ generated by $m_y m_t, m_x$. The space-time crystals in each crystal system are further classified according to their Bravais lat-

Point Group	$G_m(2)$	Generators
C_1	1	
	$11'$	m_t
D_1	m	m_x
	$m1'$	m_x, m_t
	m'	$m_x m_t$
C_2	2	R_π
	$21'$	R_π, m_t
	$2'$	$R_\pi m_t$
D_2	$mm2$	m_x, m_y
	$mm21'$	m_x, m_y, m_t
	$m'm2'$	$m_x m_t, m_y$
	$m'm'2$	$m_x m_t, m_y m_t$
C_3	3	$R_{2\pi/3}$
	$31'$	$R_{2\pi/3}, m_t$
D_3	$3m$	$R_{2\pi/3}, m_x$
	$3m1'$	$R_{2\pi/3}, m_x, m_t$
	$3m'$	$R_{2\pi/3}, m_x m_t$
C_4	4	$R_{\pi/2}$
	$41'$	$R_{\pi/2}, m_t$
	$4'$	$R_{\pi/2} m_t$
D_4	$4mm$	$R_{\pi/2}, m_x$
	$4mm1'$	$R_{\pi/2}, m_x, m_t$
	$4'm'm$	$R_{\pi/2} m_t, m_x$
	$4m'm'$	$R_{\pi/2}, m_x m_t$
C_6	6	$R_{\pi/3}$
	$61'$	$R_{\pi/3}, m_t$
	$6'$	$R_{\pi/3} m_t$
D_6	$6mm$	$R_{\pi/3}, m_x$
	$6mm1'$	$R_{\pi/3}, m_x, m_t$
	$6'm'm$	$R_{\pi/3} m_t, m_x$
	$6m'm'$	$R_{\pi/3}, m_x m_t$

TABLE I. The magnetic point groups in 2 dimensions and their relations to the usual 10 2-dimensional point groups. In the third column, the symmetry generator for each MGP is listed. The symmetry operation m_t stands for the time reversal, R_θ represents rotation in the $x - y$ plane through the angle θ , m_x is the reflection in the x direction and m_y is the reflection along the y direction.

tices. The main difference from the 3D cases occurs in the orthogonal space-time crystal systems, which contains 5 Bravais lattices rather than 4 in its static counterpart. As illustrated in Fig. 6, there are two base-centered lattices. In the t-base-centered lattice, the additional point locates at the center of the face in the $x - y$ plane, while the additional point in r-base-centered lattice is at the center of the face in the $y - t$ plane.

Crystal System	MP Group	Bravais Lattice	$G(2, 1)$
Triclinic	$1, 2'$	Primitive	2
T-Monoclinic	$11', 2, 21'$	Primitive	8
		Centered	5
R-Monoclinic	$m, m', m'm2'$	Primitive	8
		Centered	5
Orthorhombic	$mm2, m'm'2, mm21', m1'$	Primitive	68
		T-Base-Centered	15
		R-Base-Centered	22
		Face-Centered	7
Tetragonal	$4, 41', 4', 4mm, 4mm1', 4'm'm, 4m'm'$	Primitive	49
		Body-Centered	19
Trigonal	$3, 6', 3m, 3m', 6'm'm$	Primitive	18
		Rhombohedral	7
Hexagonal	$6, 61', 31', 6mm, 6m'm', 6mm1', 3m1'$	Primitive	27

TABLE II. Summary of the space-group classification in 2+1 dimensions. There are 7 space-time crystal systems and 14 space-time Bravais lattices (The primitive trigonal lattice is the same as the primitive hexagonal lattice). The 31 magnetic point groups are uniquely assigned to the 7 crystal systems, as listed in the second column. In the fourth column, we list the number of the space-time groups within each Bravais lattice.

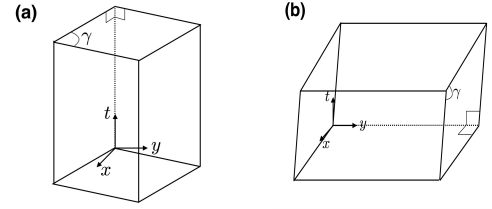


FIG. 5. The two kinds of monoclinic crystal systems in the space-time group classification. (a) The t-monoclinic lattice. One lattice basis vector is along the temporal direction and perpendicular to the other two basis vectors which are not orthogonal. The angle γ is not $\pi/2$. The maximal MPG is $21'$. (b) The r-monoclinic lattice. One lattice basis vector is along purely spatial direction and perpendicular to the other two basis vectors which are not orthogonal. The maximal MPG is $m'm2$.

In the following, we will discuss the space-time group classification of each crystal system in detail and enumerate all the space-time groups $G(2, 1)$.

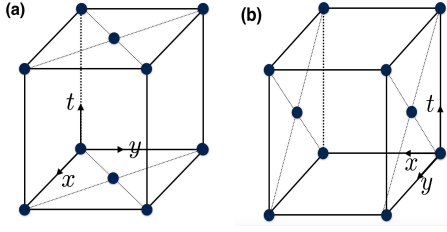


FIG. 6. The two kinds of base-centered Bravais lattices in the space-time orthorhombic crystal system. (a) The t-base-centered lattice. The additional point is at the center of the spatial face. (b) The r-base-centered lattice. The additional point is at the center of the face containing temporal axis.

1. Triclinic Crystal System

The triclinic crystal system contains the simplest space-time lattice. There is no constraint on the 3 lattice basis vectors $a_1 \sim a_3$ and thus only Bravais lattice is involved in this crystal system. The maximal symmetry of the lattices in this crystal system is described by the MGP $2'$, generated by the space-time inversion $R_\pi m_t$. There are two MGPs assigned to crystal system, the trivial one 1 and the MPG $2'$. The combination of each of two MGPs and the primitive lattice leads to two ACCs. Both of the first cohomology groups of two ACCs are identities, indicating that there is no non-trivial group extension of the primitive triclinic lattice by MGPs. In consequence, each ACC only contains one type of space-time group, and there is no nonsymmorphic space-time group in the triclinic crystal system. Physically, the combination of $R_\pi m_t$ with any translations is also space-time inversion but with a shifted origin. The two types of space-time groups are listed in Table. III.

Primitive Triclinic		
ACC NO. 1: $1P$		
$G_m(2) = 1, \quad H^1(1, T(3)/P) = \mathbb{I}$		
$P1$		1
ACC NO. 2: $2'P$		
$G_m(2) = 2', \quad H^1(2', T(3)/P) = \mathbb{I}$		
$P\bar{1} \mid R_\pi m_t$		1

TABLE III. Space-time groups in 2+1 dimensions for oblique lattices.

2. T-Monoclinic and R-Monoclinic Crystal Systems

Before classifying the space-time groups in the monoclinic crystal systems, we briefly review the conventional monoclinic lattice here. In 3 dimensions, the monoclinic lattice has a unique direction that either is a twofold rotation axis or perpendicular to a mirror plane. There are two kinds of Bravais lattices. The first is the primitive

monoclinic, hosting 8 space groups. The second is the base-centered monoclinic with an addition point at the center of the face parallel to the unique direction. There are 5 space groups in the base-centered monoclinic lattice.

In the dynamic space-time lattices considered here, however, there are two types of monoclinic lattices. This arises from the speciality of the time direction, which requires that the unique direction in the monoclinic lattice is either purely temporal or purely spatial, corresponding to the t-monoclinic crystal system and the r-monoclinic crystal system, respectively. Each of them contains two Bravais lattices and 13 space-time groups.

In the t-monoclinic crystal system, the unique direction coincides with the temporal direction. The maximal symmetry of a general t-monoclinic lattice is described by the MGP $21'$ generated by the spatial two-fold rotation R_π and time reversal m_t . Three MGP, 2, $11'$ and $21'$, are assigned to this crystal system, and therefore the t-monoclinic crystal system contains three GCCs. Although the MPG $2'$ also leave the lattices in this crystal system invariant, it also acts on the triclinic lattices with lower symmetry and thus is excluded here.

The primitive lattice basis vectors are

$$\begin{aligned} a_1^P &= (x_1, y_1, 0) \\ a_2^P &= (x_2, y_2, 0) \\ a_3^P &= (0, 0, t_0). \end{aligned} \quad (\text{A5})$$

Among the three vectors, a_1 and a_2 generally mix space and time, while a_3 is purely temporal. The combination of the three MGPs and the primitive Bravais lattice leads to three ACCs. Unlike the 2 ACCs in the triclinic crystal system, here we have nontrivial group extension of the lattice by the MGPs, and each ACC contains more than one space-time group. For example, let us look at the ACC No. 3 $2P$ and consider the interplay between the MGP 2 ($\{I, R_\pi\}$) and the translations in Eq. A5. There are two space-time groups $P112$ and $P112_1$ in this arithmetic crystal class. The space-time group $P112$ is the semi-direct product of the MGP and the lattice described by Eq. A5. The space-time group $P112_1$ extends $P112$ by replacing the 2-fold rotational symmetry R_π with the 2-fold time-screw rotational symmetry $R_\pi T_t^{1/2}$. The space-time group is nonsymmorphic because $R_\pi T_t^{1/2}$ does not fix any space-time coordinate. Mathematically, the first cohomology group of the ACC $2P$ is isomorphic to \mathbb{Z}_2 with the following generator,

$$c_1^{(3)}(R_\pi) = \frac{1}{2} a_3^P \quad (\text{A6})$$

where the superscript of c stands for the ACC id and the subscript of c labels the generator of the cohomology group. This result indicates that there are two space-time groups within the ACC $2P$.

On the other hand, $H^1(11', T(3)/P)$ is \mathbb{Z}_2^2 with the

generators,

$$\begin{aligned} c_1^{(4)}(m_t) &= \frac{1}{2}a_1^P \\ c_2^{(4)}(m_t) &= \frac{1}{2}a_2^P \end{aligned} \quad (\text{A7})$$

There are four elements in this cohomology group, in principle corresponding to four types of space-time groups. However, it turns out that c_1 , c_2 and $c_1 \cdot c_2$ can be related each other under the second equivalence relation, and there are only two distinct types of space-time groups. Physically, the symmetry operation of the time reversal with gliding along a_1^P , a_2^P and $a_1^P + a_2^P$ are identified. For the ACC 21' P , the first cohomology group is \mathbb{Z}_2^3 , and the three generators are

$$\begin{aligned} c_1^{(5)}(R_\pi) &= a_3^P/2, \quad c_1^{(5)}(m_t) = 0 \\ c_2^{(5)}(R_\pi) &= 0, \quad c_1^{(5)}(m_t) = \frac{1}{2}a_1^P \\ c_3^{(5)}(R_\pi) &= 0, \quad c_3^{(5)}(m_t) = \frac{1}{2}a_2^P \end{aligned} \quad (\text{A8})$$

There are eight group elements, but only four of them are inequivalent due to the similar reason as before. The total 8 types of the space-time groups within the primitive monoclinic lattice is listed in Table. IV.

Primitive T-Monoclinic		
ACC NO. 3: 2P		
$G_m(2) = 2, \quad H^1(2, T(3)/P) = \mathbb{Z}_2$		
P112	R_π	1
P112 ₁	$R_\pi T_t^{1/2}$	c_1
ACC NO. 4: 11'P		
$G_m(2) = 11', \quad H^1(11', T(3)/P) = \mathbb{Z}_2^2$		
P11m	m_t	1
P11c	g_t	c_1
ACC NO. 5: 21'P		
$G_m(2) = 21', \quad H^1(21', T(3)/P) = \mathbb{Z}_2^3$		
P112/m	R_π, m_t	1
P112 ₁ /m	$R_\pi T_t^{1/2}, m_t$	c_1
P112/c	R_π, g_t	c_2
P112 ₁ /c	$R_\pi T_t^{1/2}, g_t$	$c_1 \cdot c_2$

TABLE IV. Space-time groups in 2+1 dimensions for primitive t-monoclinic lattices. The action of the glide reflection g_t on the coordinate is $g_t(x, y, t) = (x, y, -t) + a_1^P/2$, and $T_t^{1/n}$ is the temporal fractional translation so that $T_t^{1/n}(x, y, t) = (x, y, t) + a_3^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the t-monoclinic crystal system defined in Eq. A5.

For the base-centered Bravais lattice, $a_1 \sim a_3$ are

$$\begin{aligned} a_1^B &= (x_1, y_1, 0) \\ a_2^B &= (x_2, y_2, 0) \\ a_3^B &= \frac{1}{2}(x_1, y_1, t_0). \end{aligned} \quad (\text{A9})$$

Combining the Bravais lattice with the same set of MPGs leads to 3 ACCs but 5 space-time groups listed in Table. V. The key difference between the base-centered Bravais lattice and the primitive one is that, in the formal case, the combination of space-time inversion R_π with temporal translation of $a_3^P/2$ is equivalent to the usual space-time reflection with a shifted origin. As a result, the first cohomology group of the ACC 2C is trivial and that of the ACC 21' C downgrades to \mathbb{Z}_2^2 , resulting in fewer types of space-time groups in these two ACCs, while the case of the ACC 11' C is unchanged from the primitive one. The 5 types of space-time groups are listed in Table. V.

Base-Centered T-Monoclinic		
ACC NO. 6: 2C		
$G_m(2) = 2, \quad H^1(2, T(3)/C) = \mathbb{I}$		
C112	R_π	1
ACC NO. 7: 11'C		
$G_m(2) = 11', \quad H^1(11', T(3)/C) = \mathbb{Z}_2^2$		
C11m	m_t	1
C11c	g_t	c_1
ACC NO. 8: 21'C		
$G_m(2) = 21', \quad H^1(21', T(3)/C) = \mathbb{Z}_2^2$		
C112/m	R_π, m_t	1
C112/c	R_π, g_t	c_1

TABLE V. Space-times group in 2+1 dimensions for base-centered t-monoclinic lattices. The action of the glide reflection g_t on the coordinate is $g_t(x, y, t) = (x, y, -t) + a_1^P/2$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the t-monoclinic crystal system defined in Eq. A5.

As to the r-monoclinic crystal system, the unique direction is purely spatial, taken to be the x axis. Two symmetry operations m_x and $m_y m_t$ leave the lattice invariant and generates the maximal MGP $m'm_2$. As a result, three MGPs, m' , m and $m'm_2$, are assigned to this crystal system. This crystal system also has two Bravais lattices. The lattice basis for the primitive one is

$$\begin{aligned} a_1^P &= (0, y_1, t_1) \\ a_2^P &= (0, y_2, t_2) \\ a_3^P &= (x_0, 0, 0), \end{aligned} \quad (\text{A10})$$

and the basis for the based-centered one is

$$\begin{aligned} a_1^B &= (0, y_1, t_1) \\ a_2^B &= (0, y_2, t_2) \\ a_3^B &= \frac{1}{2}(x_0, y_1, t_2). \end{aligned} \quad (\text{A11})$$

The classification of the space-time groups for the r-monoclinic crystal system is completely in parallel with the t-monoclinic crystal system, and is not repeated here. The resulting 6 ACCs, the corresponding cohomology

groups and 15 distinct types of space-time groups are listed in Table. VI for the primitive lattice and Table. VII for the centered lattice.

Primitive R-Monoclinic		
ACC NO. 9: $m'P$		
$G_m(2) = m', \quad H^1(m', T(3)/P) = \mathbb{Z}_2$		
$P211$	$m_y m_t$	1
$P2_111$	$m_y m_t T_x^{1/2}$	c_1
ACC NO. 10: mP		
$G_m(2) = m, \quad H^1(m, T(3)/P) = \mathbb{Z}_2^2$		
$Pm11$	m_x	1
$Pc11$	g_x	c_1
ACC NO. 11: $m'm2'P$		
$G_m(2) = m'm2', \quad H^1(m'm2', T(3)/P) = \mathbb{Z}_2^3$		
$P2/m11$	$m_x, m_y m_t$	1
$P2_1/m11$	$m_x, m_y m_t T_x^{1/2}$	c_1
$P2/c11$	$g_x, m_y m_t$	c_2
$P2_1/c11$	$g_x, m_y m_t T_x^{1/2}$	$c_1 \cdot c_2$

TABLE VI. Space-time groups in 2+1 dimensions for the primitive r-monoclinic crystal system. The action of the glide reflection g_x on the coordinate is $g_x(x, y, t) = (-x, y, t) + a_1^p/2$, and $T_x^{1/n}$ is the spatial fractional translation so that $T_x^{1/n}(x, y, t) = (x, y, t) + a_3^p/n$, where $a_1^p \sim a_3^p$ are the primitive lattice basis for r-monoclinic crystal system defined in Eq. A10.

Base-Centered R-Monoclinic		
ACC NO. 12: $m'C$		
$G_m(2) = m', \quad H^1(m', T(3)/C) = \mathbb{I}$		
$C211$	$m_y m_t$	1
ACC NO. 13: mC		
$G_m(2) = m, \quad H^1(m, T(3)/C) = \mathbb{Z}_2^2$		
$Cm11$	m_x	1
$Cc11$	g_x	c_1
ACC NO. 14: $m'm2'C$		
$G_m(2) = m'm2', \quad H^1(m'm2'C, T(3)/C) = \mathbb{Z}_2^2$		
$C2/m11$	$m_x, m_y m_t$	1
$C2/c11$	$g_x, m_y m_t$	c_1

TABLE VII. Space-time groups in 2+1 dimensions for base-centered r-monoclinic lattices. The action of the glide reflection g_x on the coordinate is $g_x(x, y, t) = (-x, y, t) + a_1^p/2$, where $a_1^p \sim a_3^p$ are the primitive lattice basis for r-monoclinic crystal system defined in Eq. A10.

3. Orthorhombic Crystal System

The space-time orthorhombic crystal system has at least two reflection/glide planes, and therefore requires three mutually orthogonal primitive lattice basis vectors. The fact that the space-time group is subgroup of $E_2 \otimes E_1$ requires that one of the three axes is purely temporal. The relevant symmetries are three reflections, m_x , m_y and m_t . The maximal symmetry of a general lattice within this crystal system is described by the MPG $mm21'$. Four MGPs, $m'm'2$, $mm2$, $m1'$ and $mm21'$, are assigned to this crystal system. In order to preserve all the MPGs, the linear map ρ in the second equivalence relation is restricted in the $x - y$ plane.

The lattice basis vectors of the primitive Bravais lattice are along the three orthogonal axes,

$$\begin{aligned} a_1^P &= (x_0, 0, 0) \\ a_2^P &= (0, y_0, 0) \\ a_3^P &= (0, 0, t_0) \end{aligned} \quad (A12)$$

The combination of the Bravais lattice and the 4 MPGs leads to 4 ACCs and 68 distinct space-time groups listed in Table. VIII and Table. IX.

The first cohomology group of the ACC $m'm'2P$ is \mathbb{Z}_2^3 with the following 3 generators,

$$\begin{aligned} c_1^{(15)}(m_x m_t) &= \frac{1}{2} a_3^P, & c_1^{(15)}(m_y m_t) &= 0 \\ c_2^{(15)}(m_x m_t) &= \frac{1}{2} a_2^P, & c_2^{(15)}(m_y m_t) &= 0 \\ c_3^{(15)}(m_x m_t) &= 0, & c_3^{(15)}(m_y m_t) &= \frac{1}{2} a_1^P \end{aligned} \quad (A13)$$

Since c_2 and c_3 are related by $\rho = R_{\pi/2}$, 6 of total 8 group elements are inequivalent as space-time group types, listed in the third column of Table. VIII. For the ACC $mm2P$, $H^1(mm2, T(3)/P) = \mathbb{Z}_2^4$ with the 4 generators reading,

$$\begin{aligned} c_1^{(16)}(m_x) &= \frac{1}{2} a_2^P, & c_1^{(16)}(m_y) &= 0 \\ c_2^{(16)}(m_x) &= \frac{1}{2} a_3^P, & c_2^{(16)}(m_y) &= 0 \\ c_3^{(16)}(m_x) &= 0, & c_3^{(16)}(m_y) &= \frac{1}{2} a_3^P \\ c_4^{(16)}(m_x) &= 0, & c_4^{(16)}(m_y) &= \frac{1}{2} a_1^P \end{aligned} \quad (A14)$$

Given the equivalence condition ρ , c_1 is related to c_4 , and c_2 is related to c_3 . Therefore the total 16 group elements give rise to 10 distinct types of space-time groups.

For the ACC $m1'P$, $H^1(mm2, T(3)/P)$ is also \mathbb{Z}_2^4 with

Primitive Orthorhombic		
ACC No.15 : $m'm'2P$		
$G_m(2): m'm'2, H^1(m'm'2, T(3)/P) = \mathbb{Z}_2^3$		
$P222$	$m_y m_t, m_x m_t$	1
$P222_1$	$m_y m_t, m_x m_t T_t^{1/2}$	c_1
$P2_122$	$m_y m_t T_x^{1/2}, m_x m_t$	c_2
$P2_12_12$	$m_y m_t T_x^{1/2}, m_x m_t T_y^{1/2}$	$c_2 \cdot c_3$
$P2_122_1$	$m_y m_t T_x^{1/2}, m_x m_t T_t^{1/2}$	$c_1 \cdot c_2$
$P2_12_12_1$	$m_y m_t T_x^{1/2}, m_x m_t T_y^{1/2} T_t^{1/2}$	$c_1 \cdot c_2 \cdot c_3$
ACC No.16 : $mm2P$		
$G_m(2): mm2, H^1(mm2, T(3)/P) = \mathbb{Z}_2^4$		
$Pmm2$	m_x, m_y	1
$Pmc2_1$	$m_x, m_y T_t^{1/2}$	c_3
$Pma2$	$m_x, m_y T_x^{1/2}$	c_4
$Pcc2$	$m_x T_t^{1/2}, m_y T_t^{1/2}$	$c_2 \cdot c_3$
$Pca2_1$	$m_x T_t^{1/2}, m_y T_x^{1/2}$	$c_2 \cdot c_4$
$Pmn2_1$	$m_x, m_y T_x^{1/2} T_t^{1/2}$	$c_3 \cdot c_4$
$Pba2$	$m_x T_y^{1/2}, m_y T_x^{1/2}$	$c_1 \cdot c_4$
$Pnc2$	$m_x T_y^{1/2} T_t^{1/2}, m_y T_t^{1/2}$	$c_1 \cdot c_2 \cdot c_3$
$Pna2_1$	$m_x T_y^{1/2} T_t^{1/2}, m_y T_x^{1/2}$	$c_1 \cdot c_2 \cdot c_4$
$Pnn2$	$m_x T_y^{1/2} T_t^{1/2}, m_y T_x^{1/2} T_t^{1/2}$	$\prod_{i=1}^4 c_i$
ACC No.17 : $m1'P$		
$G_m(2): m1', H^1(m1', T(3)/P) = \mathbb{Z}_2^4$		
$P2mm$	m_y, m_t	1
$P2_1ma$	$m_y, m_t T_x^{1/2}$	c_1
$P2_1am$	$m_y T_x^{1/2}, m_t$	c_2
$P2mb$	$m_y, m_t T_y^{1/2}$	c_3
$P2cm$	$m_y T_t^{1/2}, m_t$	c_4
$P2aa$	$m_y T_x^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_2$
$P2_1ab$	$m_y T_x^{1/2}, m_t T_y^{1/2}$	$c_2 \cdot c_3$
$P2_1ca$	$m_y T_t^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_4$
$P2_1mn$	$m_y, m_t T_x^{1/2} T_y^{1/2}$	$c_1 \cdot c_3$
$P2_1nm$	$m_y T_x^{1/2} T_t^{1/2}, m_t$	$c_2 \cdot c_4$
$P2cb$	$m_y T_t^{1/2}, m_t T_y^{1/2}$	$c_3 \cdot c_4$
$P2na$	$m_y T_x^{1/2} T_t^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_2 \cdot c_4$
$P2an$	$m_y T_x^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$c_1 \cdot c_2 \cdot c_3$
$P2_1nb$	$m_y T_x^{1/2} T_t^{1/2}, m_t T_y^{1/2}$	$c_2 \cdot c_3 \cdot c_4$
$P2_1cn$	$m_y T_t^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$c_1 \cdot c_3 \cdot c_4$
$P2nn$	$m_y T_x^{1/2} T_t^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$\prod_{i=1}^4 c_i$

TABLE VIII. Space-time groups in 2+1 dimensions for primitive orthorhombic lattice. The fractional translation $T_{x,y,z}^{1/n}$ acting on the coordinate as $T_{x,y,z}^{1/n}(x, y, z) = (x, y, z) + a_{1,2,3}^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for space-time orthorhombic crystal system defined in Eq. A12.

Primitive Orthorhombic (cont.)		
ACC No.18: $mm21'P$		
$G_m(2): mm21', H^1(mm21', T(3)/P) = \mathbb{Z}_2^6$		
$Pmmm$	m_x, m_y, m_t	1
$Pmma$	$m_x, m_y, m_t T_x^{1/2}$	c_5
$Pbmm$	$m_x T_y^{1/2}, m_y, m_t$	c_1
$Pcmm$	$m_x T_t^{1/2}, m_y, m_t$	c_2
$Pbam$	$m_x T_y^{1/2}, m_y T_x^{1/2}, m_t$	$c_2 \cdot c_4$
$Pmcb$	$m_x, m_y T_t^{1/2}, m_t T_y^{1/2}$	$c_3 \cdot c_6$
$Pccm$	$m_x T_t^{1/2}, m_y T_t^{1/2}, m_t$	$c_2 \cdot c_3$
$Pmaa$	$m_x, m_y T_x^{1/2}, m_t T_x^{1/2}$	$c_4 \cdot c_5$
$Pbcm$	$m_x T_y^{1/2}, m_y T_t^{1/2}, m_t$	$c_1 \cdot c_2$
$Pmca$	$m_x, m_y T_t^{1/2}, m_t T_x^{1/2}$	$c_3 \cdot c_5$
$Pmab$	$m_x, m_y T_x^{1/2}, m_t T_b^{1/2}$	$c_4 \cdot c_6$
$Pmmn$	$m_x, m_y, m_t T_x^{1/2} T_y^{1/2}$	$c_5 \cdot c_6$
$Pnmm$	$m_x T_y^{1/2} T_t^{1/2}, m_y, m_t$	$c_1 \cdot c_2$
$Pmna$	$m_x, m_y T_x^{1/2} T_t^{1/2}, m_t T_x^{1/2}$	$c_3 \cdot c_4 \cdot c_5$
$Pbmn$	$m_x T_y^{1/2}, m_y, m_t T_x^{1/2} T_y^{1/2}$	$c_1 \cdot c_5 \cdot c_6$
$Pncm$	$m_x T_y^{1/2} T_t^{1/2}, m_y T_t^{1/2}, m_t$	$c_1 \cdot c_2 \cdot c_3$
$Pcca$	$m_x T_t^{1/2}, m_y T_t^{1/2}, m_t T_x^{1/2}$	$c_2 \cdot c_3 \cdot c_5$
$Pbaa$	$m_x T_y^{1/2}, m_y T_x^{1/2}, m_t T_x^{1/2}$	$c_2 \cdot c_4 \cdot c_5$
$Pbcb$	$m_x T_y^{1/2}, m_y T_t^{1/2}, m_t T_y^{1/2}$	$c_2 \cdot c_3 \cdot c_6$
$Pbca$	$m_x T_y^{1/2}, m_y T_t^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_3 \cdot c_5$
$Pnma$	$m_x T_y^{1/2} T_t^{1/2}, m_y, m_t T_x^{1/2}$	$c_1 \cdot c_2 \cdot c_5$
$Pbnm$	$m_x T_y^{1/2}, m_y T_x^{1/2} T_t^{1/2}, m_t$	$c_1 \cdot c_3 \cdot c_4$
$Pmcn$	$m_x, m_y T_t^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$c_3 \cdot c_5 \cdot c_6$
$Pban$	$m_x T_y^{1/2}, m_y T_x^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$c_1 \cdot c_4 \cdot c_5$
$Pncb$	$m_x T_x^{1/2} T_y^{1/2}, m_y T_t^{1/2}, m_t T_y^{1/2}$	$c_1 \cdot c_2 \cdot c_3 \cdot c_6$
$Pccn$	$m_x T_t^{1/2}, m_y T_t^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$c_2 \cdot c_3 \cdot c_5 \cdot c_6$
$Pnaa$	$m_x T_y^{1/2} T_t^{1/2}, m_y T_x^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_2 \cdot c_4 \cdot c_5$
$Pnnm$	$m_x T_y^{1/2} T_t^{1/2}, m_y T_x^{1/2} T_t^{1/2}, m_t$	$c_1 \cdot c_2 \cdot c_3 \cdot c_4$
$Pmnn$	$m_x, m_y T_x^{1/2} T_t^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$c_3 \cdot c_4 \cdot c_5 \cdot c_6$
$Pbcn$	$m_x T_y^{1/2}, m_y T_t^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$c_1 \cdot c_3 \cdot c_5 \cdot c_6$
$Pnca$	$m_x T_y^{1/2} T_t^{1/2}, m_y T_t^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_2 \cdot c_3 \cdot c_5$
$Pnab$	$m_x T_y^{1/2} T_t^{1/2}, m_y T_x^{1/2}, m_t T_y^{1/2}$	$c_1 \cdot c_2 \cdot c_4 \cdot c_6$
$Pnna$	$m_x T_y^{1/2} T_t^{1/2}, m_y T_x^{1/2} T_t^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_2 \cdot c_3 \cdot c_4 \cdot c_5$
$Pbnn$	$m_x T_y^{1/2}, m_y T_x^{1/2} T_t^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$c_1 \cdot c_3 \cdot c_4 \cdot c_5 \cdot c_6$
$Pcnn$	$m_x T_t^{1/2}, m_y T_x^{1/2} T_t^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$c_2 \cdot c_3 \cdot c_4 \cdot c_5 \cdot c_6$
$Pnnn$	$m_x T_y^{1/2} T_t^{1/2}, m_y T_x^{1/2} T_t^{1/2}, m_t T_x^{1/2} T_y^{1/2}$	$\prod_{i=1}^6 c_i$

TABLE IX. Space-time groups in 2+1 dimensions for primitive orthorhombic lattice (continue). The fractional translation $T_{x,y,z}^{1/n}$ acting on the coordinate as $T_{x,y,z}^{1/n}(x, y, z) = (x, y, z) + a_{1,2,3}^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for space-time orthorhombic crystal system defined in Eq. A12.

the 4 generators reading,

$$\begin{aligned} c_1^{(17)}(m_y) &= \frac{1}{2}a_3^P, & c_1^{(17)}(m_t) &= 0 \\ c_2^{(17)}(m_y) &= \frac{1}{2}a_1^P, & c_2^{(17)}(m_y) &= 0 \\ c_3^{(17)}(m_y) &= 0, & c_3^{(17)}(m_t) &= \frac{1}{2}a_1^P \\ c_4^{(17)}(m_y) &= 0, & c_4^{(17)}(m_t) &= \frac{1}{2}a_2^P \end{aligned} \quad (\text{A15})$$

None of these generators are related and there are 16 distinct types for space-groups for this ACC.

For the ACC $mm21'P$ with the largest symmetry in this crystal system, the cohomology group is isomorphic to \mathbb{Z}_2^6 . The 6 generators read

$$\begin{aligned} c_1^{(18)}(m_x) &= \frac{1}{2}a_2^P, & c_1^{(18)}(m_y) &= 0, & c_1^{(18)}(m_t) &= 0 \\ c_2^{(18)}(m_x) &= \frac{1}{2}a_3^P, & c_2^{(18)}(m_y) &= 0, & c_2^{(18)}(m_t) &= 0 \\ c_3^{(18)}(m_x) &= 0, & c_3^{(18)}(m_y) &= \frac{1}{2}a_3^P, & c_3^{(18)}(m_t) &= 0 \\ c_4^{(18)}(m_x) &= 0, & c_4^{(18)}(m_y) &= \frac{1}{2}a_1^P, & c_4^{(18)}(m_t) &= 0 \\ c_5^{(18)}(m_x) &= 0, & c_5^{(18)}(m_y) &= \frac{1}{2}a_3^P, & c_5^{(18)}(m_t) &= \frac{1}{2}a_1^P \\ c_6^{(18)}(m_x) &= 0, & c_6^{(18)}(m_y) &= \frac{1}{2}a_1^P, & c_6^{(18)}(m_t) &= \frac{1}{2}a_2^P \end{aligned} \quad (\text{A16})$$

Under the equivalence relation, $c_1 \sim c_4$, $c_2 \sim c_3$ and $c_5 \sim c_6$, and there are 36 distinct types of the space-time groups out of the 64 group elements.

Now we consider the base-centered Bravais lattice. As explained in Fig. 6, different from the static counterpart, there are two base-centered Bravais lattices, t-base-centered and r-base-centered, in the space-time group classification depending on whether the base face perpendicular to the time axis or not. The lattice basis vectors of the t-base-centered Bravais lattice are

$$\begin{aligned} a_1^{tB} &= \frac{1}{2}(x_0, y_0, 0) \\ a_2^{tB} &= \frac{1}{2}(x_0, -y_0, 0) \\ a_3^{tB} &= (0, 0, t_0) \end{aligned} \quad (\text{A17})$$

The combination of the lattice and 4 MPGs leads to 4 ACCs and 15 space-time groups listed in Table. X. The first cohomology group of ACC $m'm'2C$ is \mathbb{Z}_2 with the generator,

$$c_1^{(19)}(m_x m_t) = \frac{1}{2}a_3^P, c_1^{(19)}(m_y m_t) = 0. \quad (\text{A18})$$

There are two types of space-time group within this ACC. For the ACC $mm2C$, $H^1(mm2, T(3)/C) = \mathbb{Z}_2^2$ with the two generators,

$$\begin{aligned} c_1^{(20)}(m_x) &= \frac{1}{2}a_3^P, & c_1^{(20)}(m_y) &= 0 \\ c_2^{(20)}(m_x) &= 0, & c_2^{(20)}(m_y) &= \frac{1}{2}a_3^P \end{aligned} \quad (\text{A19})$$

Under the equivalent condition, $c_1 \sim c_2$ and there are three space-time types within this ACC. For the ACC $m1'C$, $H^1(m1', T(3)/C) = \mathbb{Z}_2^2$ with the two generators,

$$\begin{aligned} c_1^{(21)}(m_x) &= \frac{1}{2}a_3^P, & c_1^{(21)}(m_t) &= 0 \\ c_2^{(21)}(m_x) &= 0, & c_2^{(21)}(m_t) &= \frac{1}{2}a_1^P \end{aligned} \quad (\text{A20})$$

These two generators are not related and constitute 4 space-time group types. For the ACC $mm21'C$, $H^1(mm21', T(3)/C) = \mathbb{Z}_2^3$ with the three generators,

$$\begin{aligned} c_1^{(22)}(m_x) &= a_3^P/2, & c_1^{(22)}(m_y) &= 0, & c_1^{(22)}(m_t) &= 0 \\ c_1^{(22)}(m_x) &= 0, & c_1^{(22)}(m_y) &= a_3^P/2, & c_1^{(22)}(m_t) &= 0 \\ c_1^{(22)}(m_x) &= 0, & c_1^{(22)}(m_y) &= 0, & c_1^{(22)}(m_t) &= a_1^P/2 \end{aligned} \quad (\text{A21})$$

The generator c_1 is related to c_2 by R_π , and the three generators lead 6 types of space-time groups.

T-Base-Centered Orthorhombic		
ACC No.19: $m'm'2C$		
$G_m(2)$: $m'm'2$, $H^1(m'm'2, T(3)/C) = \mathbb{Z}_2$		
$C222$	$m_x m_t, m_y m_t$	1
$C222_1$	$m_x m_t T_t^{1/2}, m_y m_t$	c_1
ACC No.20 : $mm2C$		
$G_m(2)$: $mm2$, $H^1(mm2, T(3)/C) = \mathbb{Z}_2^2$		
$Cmm2$	m_x, m_y	1
$Cmc2_1$	$m_x, m_y T_t^{1/2}$	c_1
$Ccc2$	$m_x T_t^{1/2}, m_y T_t^{1/2}$	$c_1 \cdot c_2$
ACC No. 21 : $m1'C$		
$G_m(2)$: $m1'$, $H^1(m1', T(3)/C) = \mathbb{Z}_2^2$		
$Cm2m$	m_x, m_t	1
$Cm2a$	$m_x, m_t T_x^{1/2}$	c_1
$Cc2m$	$m_x T_t^{1/2}, m_t$	c_2
$Cc2a$	$m_x T_t^{1/2}, T_x^{1/2} m_t$	$c_1 \cdot c_2$
ACC No.22 : $mm21'C$		
$G_m(2)$: $mm21'$, $H^1(mm21', T(3)/C) = \mathbb{Z}_2^3$		
$Cmmm$	m_x, m_y, m_t	1
$Cmcm$	$m_x, m_y T_t^{1/2}, m_t$	c_2
$Cmme$	$m_x, m_y, T_x^{1/2} m_t$	c_3
$Cmca$	$m_x, m_y T_t^{1/2}, m_t T_x^{1/2}$	$c_2 \cdot c_3$
$Cccm$	$m_x T_t^{1/2}, m_y T_t^{1/2}, m_t$	$c_1 \cdot c_2$
$Ccce$	$m_x T_t^{1/2}, m_y T_t^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_2 \cdot c_3$

TABLE X. Space-time groups in 2+1 dimensions for t-base-centered orthorhombic lattice. The fractional translation $T_{x,y,z}^{1/n}$ acts on the coordinate as $T_{x,y,t}^{1/n}(x, y, t) = (x, y, t) + a_{1,2,3}^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the space-time orthorhombic crystal system defined in Eq. A12.

On the other hand, the lattice basis vectors of r-base-

centered Bravais lattice take the following form,

$$\begin{aligned} a_1^{rB} &= (x_0, 0, 0) \\ a_2^{rB} &= \frac{1}{2}(0, y_0, t_0) \\ a_3^{rB} &= \frac{1}{2}(0, -y_0, t_0) \end{aligned} \quad (\text{A22})$$

There are 5 ACCs and 22 space-time groups, as listed in Table. XI. The MPG $m1'$ has two different ways acting on the lattice, depending on the orientation of the reflection planes and the base face. In the ACC $m_x 1' A$, the base face is within the reflection plane, while it is not in $m_y 1' A$. In the r-base-centered Bravais lattice, x and y direction is not equivalent. In consequence, for each of the 5 ACCs, none of the generators of the first cohomology group are equivalent, and the number of space-time group within each ACC is the same as the order of $H^1(G_m, T(3)/A)$. The generators of each $H^1(G_m, T/A)$ are already listed in Table. XI and not repeated here.

The face-centered orthorhombic lattice is obtained by including lattice points on each face of the primitive lattice, and the lattice basis are,

$$\begin{aligned} a_1^F &= \frac{1}{2}(x_0, y_0, 0) \\ a_2^F &= \frac{1}{2}(x_0, 0, t_0) \\ a_3^F &= \frac{1}{2}(0, y_0, t_0) \end{aligned} \quad (\text{A23})$$

The 4 ACCs and 7 space-time groups obtained by combining the translations generated by Eq. A23 and the 4 MPGs are listed in Table. XII. Each generators of the 4 cohomology groups are nonequivalent.

The body-centered orthorhombic lattice has one additional lattice point located at the center of the primitive Bravais lattice, and the corresponding lattice basis vectors are,

$$\begin{aligned} a_1^I &= \frac{1}{2}(-x_0, y_0, t_0) \\ a_2^I &= \frac{1}{2}(x_0, -y_0, t_0) \\ a_3^I &= \frac{1}{2}(x_0, y_0, -t_0) \end{aligned} \quad (\text{A24})$$

The 4 ACCs and 16 space-time groups are listed in Table. XIII. For the ACC $mm2I$, the cohomology group is \mathbb{Z}_2^2 with the two generators

$$\begin{aligned} c_1^{(33)}(m_x) &= a_2^P/2, c_1^{(33)}(m_y) = 0 \\ c_2^{(33)}(m_x) &= 0, c_2^{(33)}(m_y) = a_1^P/2 \end{aligned} \quad (\text{A25})$$

Under the equivalent relation, c_1 is related to c_2 and there are 3 space-time types. For the ACC $mm21'I$, the cohomology group is \mathbb{Z}_2^2 . The three generators are

$$\begin{aligned} c_1^{(35)}(m_x) &= a_2^P/2, c_1^{(35)}(m_y) = 0, c_1^{(35)}(m_t) = 0 \\ c_2^{(35)}(m_x) &= 0, c_2^{(35)}(m_y) = a_1^P/2, c_2^{(35)}(m_t) = 0 \\ c_3^{(35)}(m_x) &= 0, c_3^{(35)}(m_y) = 0, c_3^{(35)}(m_t) = a_1^P/2. \end{aligned} \quad (\text{A26})$$

R-Base-Centered Orthorhombic		
ACC No.23: $m' m' 2A$		
$G_m(2)$: $m' m' 2$, $H^1(m' m' 2, T(3)/A) = \mathbb{Z}_2$		
$A222$	$m_y m_t, m_x m_t$	1
$A2_1 22$	$m_y m_t, T_x^{1/2} m_x m_t$	c_1
ACC No.24 : $m_y 1' A$		
$G_m(2)$: $m1'$, $H^1(m1', T(3)/A) = \mathbb{Z}_2^2$		
$A2mm$	m_y, m_t	1
$A2_1 ma$	$m_y, m_t T_x^{1/2}$	c_2
$A2_1 am$	$m_y T_x^{1/2}, m_t$	c_1
$A2aa$	$m_y T_x^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_2$
ACC No.25 : $m_x 1' A$		
$G_m(2)$: $m1'$, $H^1(m1', T(3)/A) = \mathbb{Z}_2^2$		
$Am2m$	m_x, m_t	1
$Ab2m$	$m_x T_y^{1/2}, m_t$	c_1
$Am2a$	$m_x, m_t T_x^{1/2}$	c_2
$Ac2a$	$m_x T_x^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_2$
ACC No.26 : $mm2A$		
$G_m(2)$: $mm2$, $H^1(mm2A, T(3)/A) = \mathbb{Z}_2^2$		
$Amm2$	m_x, m_y	1
$Abm2$	$m_x T_y^{1/2}, m_y$	c_1
$Ama2$	$m_x, m_y T_x^{1/2}$	c_2
$Aba2$	$m_x T_y^{1/2}, m_y T_x^{1/2}$	$c_1 \cdot c_2$
ACC No.27 : $mm21' A$		
$G_m(2)$: $mm21'$, $H^1(mm21' A, T(3)/A) = \mathbb{Z}_2^3$		
$Ammm$	m_x, m_y, m_t	1
$Aemm$	$m_x T_y^{1/2}, m_y, m_t$	c_1
$Amam$	$m_x, m_y T_x^{1/2}, m_t$	c_2
$Amma$	$m_x, m_y, m_t T_x^{1/2}$	c_3
$Abma$	$m_x T_y^{1/2}, m_y, m_t T_x^{1/2}$	$c_1 \cdot c_3$
$Abam$	$m_x T_y^{1/2}, m_y T_x^{1/2}, m_t$	$c_1 \cdot c_2$
$Amaa$	$m_x, m_y T_x^{1/2}, m_t T_x^{1/2}$	$c_2 \cdot c_3$
$Aeaa$	$m_x T_y^{1/2}, m_y T_x^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_2 \cdot c_3$

TABLE XI. Space-time group in 2+1 dimension for r-base-centered orthorhombic lattice. The fractional translation $T_{x,y,z}^{1/n}$ acting on the coordinate as $T_{x,y,z}^{1/n}(x, y, z) = (x, y, z) + a_{1,2,3}^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the orthorhombic crystal system defined in Eq. A12.

Among the three generators, $c_1 \sim c_2$, and there are 6 types of space-groups.

4. Tetragonal Crystal System

The tetragonal crystal system has 4-fold rotational symmetry. In the space-time group classification, due to the absence of the rotational symmetry mixing space and time, the rotation plane has to be purely spatial. In

Face-Centered Orthorhombic		
ACC No.28 : $m'm'2F$		
$G_m(2)$: $mm2F$, $H^1(m'm'2, T(3)/F) = \mathbb{I}$		
$F222$	$m_x m_t, m_y m_t$	1
ACC No.29 : $mm2F$		
$G_m(2)$: $mm2$, $H^1(mm2, T(3)/F) = \mathbb{Z}_2$		
$Fmm2$	m_x, m_y	1
$Fdd2$	$m_x T_y^{1/4} T_t^{1/4}, m_y T_x^{1/4} T_t^{1/4}$	c_1
ACC No.30 : $m1'F$		
$G_m(2)$: $m1'$, $H^1(m1', T(3)/F) = \mathbb{Z}_2$		
$F2mm$	m_y, m_t	1
$F2dd$	$m_y T_x^{1/4} T_t^{1/4}, m_t T_x^{1/4} T_y^{1/4}$	c_1
ACC No.31 : $mm21'F$		
$G_m(2)$: $mm21'$, $H^1(mm21', T(3)/F) = \mathbb{Z}_2$		
$Fmmm$	m_x, m_y, m_t	1
$Fddd$	$m_x T_y^{1/4} T_t^{1/4}, m_y T_x^{1/4} T_t^{1/4}, m_t T_x^{1/4} T_y^{1/4}$	c_1

TABLE XII. Space-time groups in 2+1 dimensions for face-centered orthorhombic lattices. The fractional translation $T_{x,y,t}^{1/n}$ acting on the coordinate as $T_{x,y,t}^{1/n}(x, y, t) = (x, y, t) + a_{1,2,3}^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the space-time orthorhombic crystal class defined in Eq. A12.

consequence, there is one to one correspondence between the space groups and space-time groups in this crystal system. The relevant MPG symmetry operations are 4-fold rotation $R_{\pi/2}$, spatial reflection m_x and time reversal m_t . Seven MPGs constructed from these symmetry operations are assigned to this crystal system. They are 4, 4', 41', 4mm, 4mm1', 4'm'm and 4m'm'.

There are two Bravais lattices. The primitive one has the following the basis vectors,

$$\begin{aligned} a_1^P &= (u, 0, 0) \\ a_2^P &= (0, u, 0) \\ a_3^P &= (0, 0, t_0) \end{aligned} \quad (A27)$$

The semi-direct products of the 7 MGPs and the translation generated by Eq. A27 lead to 8 arithmetic crystal classes and total 49 space-time groups listed in Table. XIV and XV.

The reason 7 MGPs lead to 8 arithmetic crystal class is that the MGP 4'm'm can act on the Bravais lattice in two different ways depending on the direction of the reflection line. In the first case, the reflection line is parallel to a_1^P or a_2^P , and the matrix representations of the two generators $R_{\pi/2}m_t$ and m , in term of the three lattice vectors, are

$$\mathcal{D}_1(R_{\pi/2}m_t) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \mathcal{D}_1(m) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (A28)$$

Body-Centered Orthorhombic		
ACC No.32 : $m'm'2I$		
$G_m(2)$: $m'm'2$, $H^1(m'm'2, T(3)/I) = \mathbb{Z}_2$		
$I222$	$m_x m_t, m_y m_t$	1
$I212121$	$m_x m_t T_y^{1/2} T_t^{1/2}, m_y m_t T_x^{1/2} T_y^{1/2}$	c_1
ACC No.33 : $mm2I$		
$G_m(2)$: $mm2$, $H^1(mm2, T(3)/I) = \mathbb{Z}_2^2$		
$Imm2$	m_x, m_y	1
$Ima2$	$m_x, m_y T_x^{1/2}$	c_1
$Iba2$	$m_x T_y^{1/2}, m_y T_x^{1/2}$	$c_1 \cdot c_2$
ACC No.34 : $m1'I$		
$G_m(2)$: $m1'$, $H^1(mm2, T(3)/I) = \mathbb{Z}_2^2$		
$I2mm$	m_y, m_t	1
$I2mb$	$m_y, m_t T_y^{1/2}$	c_2
$I2cm$	$m_y T_t^{1/2}, m_t$	c_1
$I2bc$	$m_y T_t^{1/2}, m_t T_y^{1/2}$	$c_1 \cdot c_2$
ACC No.35 : $mm21'I$		
$G_m(2)$: $mm21'$, $H^1(mm21', T(3)/I) = \mathbb{Z}_2^3$		
$Immm$	m_x, m_y, m_t	1
$Imma$	$m_x, m_y, m_t T_x^{1/2}$	c_3
$Ibmm$	$m_x T_y^{1/2}, m_y, m_t$	c_1
$Ibam$	$m_x T_y^{1/2}, m_y T_x^{1/2}, m_t$	$c_1 \cdot c_2$
$Imcb$	$m_x, m_y T_t^{1/2}, m_t T_y^{1/2}$	$c_2 \cdot c_3$
$Ibca$	$m_x T_y^{1/2}, m_y T_t^{1/2}, m_t T_x^{1/2}$	$c_1 \cdot c_2 \cdot c_3$

TABLE XIII. Space-time groups in 2+1 dimensions for body-centered orthorhombic lattices. The fractional translations $T_{x,y,t}^{1/n}$ act on the coordinate as $T_{x,y,t}^{1/n}(x, y, t) = (x, y, t) + a_{1,2,3}^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the space-time orthorhombic crystal system defined in Eq. A12.

In the second case, the reflection is parallel to $a_1^P \pm a_2^P$, and the matrix representation of the reflection is different from the first case,

$$\mathcal{D}_2(R_{\pi/2}m_t) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \mathcal{D}_2(m) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (A29)$$

These two cases actually belong to the same geometry crystal class but different arithmetic crystal classes.

The body-centered Bravais lattice is obtained by adding one more lattice point at the body center of the primitive lattice with the following basis vectors

$$\begin{aligned} a_1^I &= \frac{1}{2}(-u, u, t_0) \\ a_2^I &= \frac{1}{2}(u, -u, t_0) \\ a_3^I &= \frac{1}{2}(u, u, -t_0), \end{aligned} \quad (A30)$$

Due to the same reason for the primitive lattice, there

Primitive Tetragonal		
ACC No.36 : 4P		
$G_m(2) : 4, H^1(4, T(3)/P) = \mathbb{Z}_4$		
$P4$	$R_{\pi/2}$	1
$P4_1$	$R_{\pi/2}T_t^{1/4}$	c_1
$P4_2$	$R_{\pi/2}T_t^{1/2}$	c_1^2
$P4_3$	$R_{\pi/2}T_t^{3/4}$	c_1^3
ACC No.37 : 4'P		
$G_m(2) : 4, H^1(4', T(3)/P) = \mathbb{I}$		
$P\bar{4}$	$R_{\pi/2}m_t$	1
ACC No.38 : 41'P		
$G_m(2) : 41', H^1(41', T(3)/P) = \mathbb{Z}_2^2$		
$P4/m$	$R_{\pi/2}, m_t$	1
$P4_2/m$	$R_{\pi/2}T_t^{1/2}, m_t$	c_1
$P4/n$	$R_{\pi/2}, m_tT_x^{1/2}T_y^{1/2}$	c_2
$P4_2/n$	$R_{\pi/2}T_t^{1/2}, m_tT_x^{1/2}T_y^{1/2}$	$c_1 \cdot c_2$
ACC No.39 : 4m'm'P		
$G_m(2) : 4m'm', H^1(4m'm', T(3)/P) = \mathbb{Z}_4 \otimes \mathbb{Z}_2$		
$P422$	$R_{\pi/2}, m_xm_t$	1
$P42_12$	$R_{\pi/2}, m_xm_tT_x^{1/2}T_y^{1/2}$	c_1
$P4_122$	$R_{\pi/2}T_t^{1/4}, m_xm_t$	c_2
$P4_222$	$R_{\pi/2}T_t^{1/2}, m_xm_t$	c_2^2
$P4_322$	$R_{\pi/2}T_t^{3/4}, m_xm_t$	c_2^3
$P4_12_12$	$R_{\pi/2}T_t^{1/4}, m_xm_tT_x^{1/2}T_y^{1/2}$	$c_1 \cdot c_2$
$P4_22_12$	$R_{\pi/2}T_t^{1/2}, m_xm_tT_x^{1/2}T_y^{1/2}$	$c_1 \cdot c_2^2$
$P4_32_12$	$R_{\pi/2}T_t^{3/4}, m_xm_tT_x^{1/2}T_y^{1/2}$	$c_1 \cdot c_2^3$

TABLE XIV. Space-time groups in 2+1 dimensions for primitive tetragonal lattices. The fractional translation $T_{x,y,t}^{1/n}$ acting on the coordinate as $T_{x,y,t}^{1/n}(x, y, t) = (x, y, t) + a_{1,2,3}^P/n$, where $a_1^P \sim a_3^P$ are the lattice basis for the space-time tetragonal crystal system defined in Eq. A27.

are also 8 arithmetic crystal classes, which is constituted of 19 space-time groups, as listed in Table. XVI. For all ACCs within the tetragonal crystal system, all the elements of the cohomology group are one-to-one corresponding to distinct space-time group types. All the generators are indicated in Table. XIV, XV and XVI.

5. Trigonal Crystal System

The symmetry operation of the trigonal crystal system contains 3-fold rotation. Since rotations mixing space and time is forbidden, the plane of the 3-fold rotation is spatial. In consequence, the space-time groups in this crystal system have one to one correspondence to the 3D trigonal space groups.

There are two Bravais lattices, the primitive trigonal lattice and the rhombohedral lattice. The primitive lat-

Primitive Tetragonal (cont.)		
ACC No.40 : 4mmP		
$G_m(2) : 4mm, H^1(4mm, T(3)/P) = \mathbb{Z}_2^3$		
$P4mm$	$R_{\pi/2}, m_x$	1
$P4_2mc$	$R_{\pi/2}T_t^{1/2}, m_x$	c_1
$P4bm$	$R_{\pi/2}, m_xT_x^{1/2}T_y^{1/2}$	c_2
$P4cc$	$R_{\pi/2}, m_xT_t^{1/2}$	c_3
$P4_2cm$	$R_{\pi/2}T_t^{1/2}, m_xT_t^{1/2}$	$c_1 \cdot c_3$
$P4nc$	$R_{\pi/2}, m_xT_x^{1/2}T_y^{1/2}T_t^{1/2}$	$c_2 \cdot c_3$
$P4_2bc$	$R_{\pi/2}T_t^{1/2}, m_xT_x^{1/2}T_y^{1/2}$	$c_1 \cdot c_2$
$P4_2nm$	$R_{\pi/2}T_t^{1/2}, m_xT_x^{1/2}T_y^{1/2}T_t^{1/2}$	$c_1 \cdot c_2 \cdot c_3$
ACC No.41 : 4'm'm'P		
$G_m(2) : 4'm'm', H^1(4'm'm', T(3)/P) = \mathbb{Z}_2^2$		
$P\bar{4}2m$	$R_{\pi/2}m_t, m_xm_t$	1
$P\bar{4}2c$	$R_{\pi/2}m_t, m_xm_tT_t^{1/2}$	c_1
$P\bar{4}2_1m$	$R_{\pi/2}m_t, m_xm_tT_x^{1/2}T_y^{1/2}$	c_2
$P\bar{4}2_1c$	$R_{\pi/2}m_t, m_xm_tT_x^{1/2}T_y^{1/2}T_t^{1/2}$	$c_1 \cdot c_2$
ACC No.42: 4'mm'P		
$G_m(2) : 4'mm', H^1(4'm'm', T(3)/P) = \mathbb{Z}_2^2$		
$P\bar{4}m2$	$R_{\pi/2}m_t, m_x$	1
$P\bar{4}c2$	$R_{\pi/2}m_t, m_xT_t^{1/2}$	c_1
$P\bar{4}b2$	$R_{\pi/2}m_t, m_xT_x^{1/2}T_y^{1/2}$	c_2
$P\bar{4}n2$	$R_{\pi/2}m_t, m_xT_x^{1/2}T_y^{1/2}T_t^{1/2}$	$c_1 \cdot c_2$
ACC No.43 : 4mm1'P		
$G_m(2) : 4mm1', H^1(4mm1', T(3)/P) = \mathbb{Z}_2^4$		
$P4/mmm$	$R_{\pi/2}, m_x, m_t$	1
$P4_2/mmc$	$R_{\pi/2}T_t^{1/2}, m_x, m_t$	c_1
$P4/mbm$	$R_{\pi/2}, m_xT_x^{1/2}T_y^{1/2}, m_t$	c_2
$P4/mcc$	$R_{\pi/2}, m_xT_t^{1/2}, m_t$	c_3
$P4/nmm$	$R_{\pi/2}, m_x, m_tT_x^{1/2}T_y^{1/2}$	c_4
$P4/nbm$	$R_{\pi/2}, m_xT_x^{1/2}T_y^{1/2}, m_tT_x^{1/2}T_y^{1/2}$	$c_2 \cdot c_4$
$P4/mnc$	$R_{\pi/2}, m_xT_x^{1/2}T_y^{1/2}T_t^{1/2}, m_t$	$c_2 \cdot c_3$
$P4/ncc$	$R_{\pi/2}, m_xT_t^{1/2}, m_tT_x^{1/2}T_y^{1/2}$	$c_3 \cdot c_4$
$P4_2/nmc$	$R_{\pi/2}T_t^{1/2}, m_x, m_tT_x^{1/2}T_y^{1/2}$	$c_1 \cdot c_4$
$P4_2/mcm$	$R_{\pi/2}T_t^{1/2}, m_xT_t^{1/2}, m_t$	$c_1 \cdot c_3$
$P4_2/mbc$	$R_{\pi/2}T_t^{1/2}, m_xT_x^{1/2}T_y^{1/2}, m_t$	$c_1 \cdot c_2$
$P4_2/nbc$	$R_{\pi/2}T_t^{1/2}, m_xT_x^{1/2}T_y^{1/2}, T_x^{1/2}T_y^{1/2}m_t$	$c_1 \cdot c_2 \cdot c_4$
$P4/nnc$	$R_{\pi/2}, m_xT_x^{1/2}T_y^{1/2}T_t^{1/2}, m_tT_x^{1/2}T_y^{1/2}$	$c_2 \cdot c_3 \cdot c_4$
$P4_2/nmm$	$R_{\pi/2}T_t^{1/2}, m_xT_x^{1/2}T_y^{1/2}T_t^{1/2}, m_t$	$c_1 \cdot c_2 \cdot c_3$
$P4_2/ncm$	$R_{\pi/2}T_t^{1/2}, m_xT_t^{1/2}, m_tT_x^{1/2}T_y^{1/2}$	$c_1 \cdot c_3 \cdot c_4$
$P4_2/nnm$	$R_{\pi/2}T_t^{1/2}, m_xT_x^{1/2}T_y^{1/2}T_t^{1/2}$ $T_x^{1/2}T_y^{1/2}m_t$	$\prod_{i=1}^4 c_i$

TABLE XV. Space-time groups in 2+1 dimensions for primitive tetragonal lattices (cont.). The fractional translation $T_{x,y,t}^{1/n}$ act on the coordinate as $T_{x,y,t}^{1/n}(x, y, t) = (x, y, t) + a_{1,2,3}^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the tetragonal crystal system defined in Eq. A27.

Body-Centered Tetragonal		
ACC No.44 : $4I$		
$G_m(2) : 4, H^1(4, T(3)/I) = \mathbb{Z}_2$		
$I4$	$R_{\pi/2}$	1
$I4_1$	$R_{\pi/2}T_t^{1/4}$	c_1
ACC No.45 : $4'I$		
$G_m(2) : 4', H^1(4', T(3)/I) = \mathbb{I}$		
$I\bar{4}$	$R_{\pi/2}m_t$	1
ACC No.46 : $41'I$		
$G_m(2) : 41', H^1(41', T(3)/I) = \mathbb{Z}_2$		
$I4/m$	$R_{\pi/2}, m_t$	1
$I4_1/a$	$R_{\pi/2}T_t^{1/4}, m_tT_x^{1/2}$	c_1
ACC No.47 : $4m'm'I$		
$G_m(2) : 4m'm', H^1(4m'm', T(3)/I) = \mathbb{Z}_2$		
$I422$	$R_{\pi/2}, m_xm_t$	1
$I4_122$	$R_{\pi/2}T_t^{1/4}, m_xm_t$	c_1
ACC No.48 : $4mmI$		
$G_m(2) : 4mm, H^1(4mm, T(3)/I) = \mathbb{Z}_2^2$		
$I4mm$	$R_{\pi/2}, m_x$	1
$I4cm$	$R_{\pi/2}, m_xT_t^{1/2}$	c_2
$I4_1cd$	$R_{\pi/2}T_t^{1/4}, m_xT_y^{1/2}$	c_1
$I4_1md$	$R_{\pi/2}T_t^{1/4}, m_xT_y^{1/2}T_t^{1/2}$	$c_1 \cdot c_2$
ACC No.49 : $4'mm'I$		
$G_m(2) : 4'mm', H^1(4'mm', T(3)/I) = \mathbb{Z}_2$		
$I\bar{4}m2$	$R_{\pi/2}m_t, m_x$	1
$I\bar{4}c2$	$R_{\pi/2}m_t, m_xT_t^{1/2}$	c_1
ACC No.50 : $4'mm'I$		
$G_m(2) : 4'mm', H^1(4'mm', T(3)/I) = \mathbb{Z}_2$		
$I\bar{4}2m$	$R_{\pi/2}m_t, m_xm_t$	1
$I\bar{4}2d$	$R_{\pi/2}m_t, m_xm_tT_x^{1/2}T_t^{1/4}$	c_1
ACC No.51 : $4mm1'I$		
$G_m(2) : 4mm1', H^1(4mm1', T(3)/I) = \mathbb{Z}_2^2$		
$I4/mmm$	$R_{\pi/2}, m_x, m_t$	1
$I4/mcm$	$R_{\pi/2}, m_xT_t^{1/2}, m_t$	c_2
$I4_1/amd$	$R_{\pi/2}T_t^{1/4}, m_xT_x^{1/2}, m_tT_x^{1/2}$	c_1
$I4_1/acd$	$R_{\pi/2}T_t^{1/4}, m_xT_x^{1/2}T_t^{1/2}, m_tT_x^{1/2}$	$c_1 \cdot c_2$

TABLE XVI. Space-time groups in 2+1 dimensions for body-centered-tetragonal lattices. The fractional translation $T_{x,y,z}^{1/n}$ act on the coordinate as $T_{x,y,t}^{1/n}(x, y, t) = (x, y, t) + a_{1,2,3}^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the space-time tetragonal crystal system defined in Eq. A27.

tice basis vectors are

$$\begin{aligned}
a_1^P &= (u, 0, 0) \\
a_2^P &= \left(-\frac{1}{2}u, \frac{\sqrt{3}}{2}u, 0\right) \\
a_3^P &= (0, 0, t_0).
\end{aligned} \tag{A31}$$

The rhombohedral lattice is obtained by including two additional lattice points trisecting the body diagonal of the primitive unit cell, with the lattice basis vectors

$$\begin{aligned}
a_1^R &= \left(-\frac{u}{2}, -\frac{\sqrt{3}u}{6}, \frac{t_0}{3}\right) \\
a_2^R &= \left(\frac{u}{2}, -\frac{\sqrt{3}u}{6}, \frac{t_0}{3}\right) \\
a_3^R &= \left(0, \frac{\sqrt{3}u}{3}, \frac{t_0}{3}\right).
\end{aligned} \tag{A32}$$

There are 5 MPGs, 3, $6'$, $3m$, $3m'$ and $6'm'm$, leaving both the primitive lattice and the rhombohedra lattice invariant and therefore assigned to the trigonal crystal system.

The combination of the 5 MPGs and the primitive lattice defined in Eq. A31 gives rise to 8 arithmetic crystal classes because the three among the 5 MGPs, $3m$, $3m'$ and $6'm'm$, have two inequivalent ways acting on the lattice according to the orientation of the reflection plane. There are 18 space-time groups falling into the 8 arithmetic crystal classes for the primitive trigonal lattice.

On the other hand, there are only 5 arithmetic crystal classes for the rhombohedra lattice as there is no ambiguity on how the MGPs act on lattice basis in Eq. A32. There are total 6 space-time groups defined on the rhombohedra lattice, as listed in Table. XVIII.

All the generators of the first cohomology groups for each of the 13 ACCs are independent, and are listed in Table. XVII and XVIII.

6. Hexagonal Crystal System

The symmetry of hexagonal crystal system involves 6-fold rotation. The rotation plane is purely spatial. This crystal system only contains the primitive Bravais lattice, generated by the following vectors

$$\begin{aligned}
a_1^P &= (u, 0, 0) \\
a_2^P &= \left(-\frac{1}{2}u, \frac{\sqrt{3}}{2}u, 0\right) \\
a_3^P &= (0, 0, t_0).
\end{aligned} \tag{A33}$$

There are 7 MGPs, 6, $61'$, $31'$, $6mm$, $6m'm'$, $6mm1'$ and $3m1'$, leaving the lattice invariant and assigned to the hexagonal crystal system. It is worth noting that the primitive hexagonal lattice is the same as the primitive trigonal lattice in Eq. A31. But the 7 MGPs assigned to the hexagonal crystal system *only* act on the primitive lattice, while the 5 MPGs assigned to the trigonal crystal system are the symmetry groups of both primitive lattice and rhombohedral lattice. The 7 MGPs act on the primitive lattice unambiguously except $3m1'$ which has two inequivalent ways acting on the primitive lattice, giving rise to 8 ACCs. All the generators of the first cohomology groups for each of the 8 ACCs are independent, resulting in 27 space-time group types listed in Table. XIX.

Primitive Trigonal		
ACC No.52 : $3P$		
$G_m(2) : 3, H^1(3, T(3)/P) = \mathbb{Z}_3$		
$P3$	$R_{2\pi/3}$	1
$P3_1$	$R_{2\pi/3} T_t^{1/3}$	c_1
$P3_2$	$R_{2\pi/3} T_t^{2/3}$	c_1^2
ACC No.53 : $6'P$		
$G_m(2) : 6', H^1(6', T(3)/P) = \mathbb{Z}_2$		
$P\bar{3}$	$R_{\pi/3} m_t$	1
ACC No.54 : $312P$		
$G_m(2) : 3m', H^1(3m', T(3)/P) = \mathbb{Z}_3$		
$P312$	$R_{2\pi/3}, m_x m_t$	1
$P3_1 12$	$R_{2\pi/3} T_t^{1/3}, m_x m_t$	c_1
$P3_2 12$	$R_{2\pi/3} T_t^{2/3}, m_x m_t$	c_1^2
ACC No.55 : $321P$		
$G_m(2) : 3m', H^1(3m', T(3)/P) = \mathbb{Z}_3$		
$P321$	$R_{2\pi/3}, m_y m_t$	1
$P3_1 21$	$R_{2\pi/3} T_t^{1/3}, m_y m_t$	c_1
$P3_2 21$	$R_{2\pi/3} T_t^{2/3}, m_y m_t$	c_1^2
ACC No.56 : $3m1P$		
$G_m(2) : 3m, H^1(3m, T(3)/P) = \mathbb{Z}_2$		
$P3m1$	$R_{2\pi/3}, m_x$	1
$P3c1$	$R_{2\pi/3}, m_x T_t^{1/2}$	c_1
ACC No.57 : $31mP$		
$G_m(2) : 3m, H^1(3m, T(3)/P) = \mathbb{Z}_2$		
$P31m$	$R_{2\pi/3}, m_y$	1
$P31c$	$R_{2\pi/3}, m_y T_t^{1/2}$	c_1
ACC No.58 : $6'm'mP$		
$G_m(2) : 6'm'm, H^1(6'm'm, T(3)/P) = \mathbb{Z}_2$		
$P\bar{3}1m$	$R_{\pi/3} m_t, m_y$	1
$P\bar{3}1c$	$R_{\pi/3} m_t, m_y T_t^{1/2}$	c_1
ACC No.59 : $6'mm'P$		
$G_m(2) : 6'm'm, H^1(6'm'm, T(3)/P) = \mathbb{Z}_2$		
$P\bar{3}m1$	$R_{\pi/3} m_t, m_x$	1
$P\bar{3}c1$	$R_{\pi/3} m_t, m_x T_t^{1/2}$	c_1

TABLE XVII. Space-time groups in 2+1 dimensions for primitive trigonal lattices. The fractional temporal translation $T_t^{1/n}$ acts on coordinate as $T_t^{1/n}(x, y, z) = (x, y, z) + a_3^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the space-time trigonal crystal system defined in Eq. A31.

Rhombohedral		
ACC No.60 : $3R$		
$G_m(2) : 3, H^1(3, T(3)/R) = \mathbb{I}$		
$R3$	$R_{2\pi/3}$	1
ACC No.61 : $6'R$		
$G_m(2) : 6', H^1(6', T(3)/R) = \mathbb{I}$		
$R\bar{3}$	$R_{\pi/3} m_t$	1
ACC No.62 : $3m'R$		
$G_m(2) : 3m', H^1(3m', T(3)/R) = \mathbb{I}$		
$R32$	$R_{2\pi/3}, m_x m_t$	1
ACC No.63 : $3mR$		
$G_m(2) : 3m, H^1(3m, T(3)/R) = \mathbb{Z}_2$		
$R3m$	$R_{2\pi/3}, m_x$	1
$R3c$	$R_{2\pi/3}, m_x T_t^{1/2}$	c_1
ACC No.64 : $6'm'mR$		
$G_m(2) : 6'm'm, H^1(6'm'm, T(3)/R) = \mathbb{Z}_2$		
$R\bar{3}m$	$R_{\pi/3} m_t, m_x$	1
$R\bar{3}c$	$R_{\pi/3} m_t, m_x T_t^{1/2}$	c_1

TABLE XVIII. Space-time groups in 2+1 dimensions for rhombohedral lattices. The fractional temporal translation $T_t^{1/n}$ acts on the space-time coordinate as $T_t^{1/n}(x, y, t) = (x, y, t) + a_3^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the trigonal crystal system defined in Eq. A31.

Primitive Hexagonal		
ACC No.65 : $6P$		
$G_m(2) : 6, H^1(6, T(3)/P) = \mathbb{Z}_6$		
$P6$	$R_{\pi/3}$	1
$P6_1$	$R_{\pi/3}T_t^{1/6}$	c_1
$P6_2$	$R_{\pi/3}T_t^{1/3}$	c_1^2
$P6_3$	$R_{\pi/3}T_t^{1/2}$	c_1^3
$P6_4$	$R_{\pi/3}T_t^{2/3}$	c_1^4
$P6_5$	$R_{\pi/3}T_t^{5/6}$	c_1^5
ACC No.66 : $31'P$		
$G_m(2) : 31', H^1(31', T(3)/P) = \mathbb{I}$		
$P\bar{6}$	$R_{2\pi/3}, m_t$	1
ACC No.67 : $61'P$		
$G_m(2) : 61', H^1(61', T(3)/P) = \mathbb{Z}_2$		
$P6/m$	$R_{\pi/3}, m_t$	1
$P6_3/m$	$R_{\pi/3}T_t^{1/2}, m_t$	c_1
ACC No.68 : $6m'm'P$		
$G_m(2) : 6m'm', H^1(6m'm', T(3)/P) = \mathbb{Z}_6$		
$P6222$	$R_{\pi/3}, m_x m_t$	1
$P6_1222$	$R_{\pi/3}T_t^{1/6}, m_x m_t$	c_1
$P6_2222$	$R_{\pi/3}T_t^{1/3}, m_x m_t$	c_1^2
$P6_3222$	$R_{\pi/3}T_t^{1/2}, m_x m_t$	c_1^3
$P6_4222$	$R_{\pi/3}T_t^{2/3}, m_x m_t$	c_1^4
$P6_5222$	$R_{\pi/3}T_t^{5/6}, m_x m_t$	c_1^5
ACC No.69 : $6mmP$		
$G_m(2) : 6mm, H^1(6mm, T(3)/P) = \mathbb{Z}_2^2$		
$P6mm$	$R_{\pi/3}, m_x$	1
$P6_3mc$	$R_{\pi/3}T_t^{1/2}, m_x$	c_1
$P6cc$	$R_{\pi/3}, m_x T_t^{1/2}$	c_2
$P6_3cm$	$R_{\pi/3}T_t^{1/2}, m_x T_t^{1/2}$	$c_1 \cdot c_2$
ACC No.70 : $3m_x1'P$		
$G_m(2) : 3m1', H^1(3m', T(3)/P) = \mathbb{Z}_2$		
$P\bar{6}m2$	$R_{2\pi/3}, m_x, m_t$	1
$P\bar{6}c2$	$R_{2\pi/3}, m_x T_t^{1/2}, m_t$	c_1
ACC No.71 : $3m_y1'P$		
$G_m(2) : 3m1', H^1(3m', T(3)/P) = \mathbb{Z}_2$		
$P\bar{6}2m$	$R_{2\pi/3}, m_y, m_t$	1
$P\bar{6}2c$	$R_{2\pi/3}, m_y T_t^{1/2}, m_t$	c_1
ACC No.72 : $6mm1'P$		
$G_m(2) : 6mm1', H^1(6mm1', T(3)/P) = \mathbb{Z}_2^2$		
$P6/mmm$	$R_{\pi/3}, m_x, m_t$	1
$P6_3/mcm$	$R_{\pi/3}T_t^{1/2}, m_x T_t^{1/2}, m_t$	c_1
$P6_3/mmc$	$R_{\pi/3}T_t^{1/2}, m_x, m_t$	c_2
$P6/mcc$	$R_{\pi/3}, m_x T_t^{1/2}, m_t$	$c_1 \cdot c_2$

TABLE XIX. Space-time groups in 2+1 dimensions for primitive hexagonal lattices. The fractional temporal translation $T_t^{1/n}$ acts on the coordinate as $T_t^{1/n}(x, y, t) = (x, y, t) + a_3^P/n$, where $a_1^P \sim a_3^P$ are the primitive lattice basis for the space-time hexagonal crystal class defined in Eq. A33.

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